Welcome back to Extensions!
Here are some things to do whilst we wait for everyone to join us…

KING'S MATHS SCHOOL

1. Dr Gizmo has invented a coin changing machine which can be used in any country in the world. No matter what the currency, the machine takes any coin, and (if possible) returns five others with the same total value as the original coin.
If we start with a single coin, can we end up (at some point in the future) with 26 coins, using only Dr Gizmo's machine?

2. There are 16 pennies on a table. Two players take it in turns to each remove between 1 and 4 pennies (inclusive). The winner is the last person who can remove any pennies. Who has a winning strategy? [Hint: try with a smaller number of starting pennies]

Intro problems

KING'S MATHS SCHOOL

1. Dr Gizmo has invented a coin changing machine which can be used in any country in the world. No matter what the currency, the machine takes any coin, and (if possible) returns five others with the same total value as the original coin.

If we start with a single coin, can we end up (at some point in the future) with 26 coins, using only Dr Gizmo's machine?

Total coin count

$1 \to 5 \to 9 \to 13 \to 17 \to 21 \to 25 \to 29$

Sequence: "$4n-3$"

Odd number of coins!

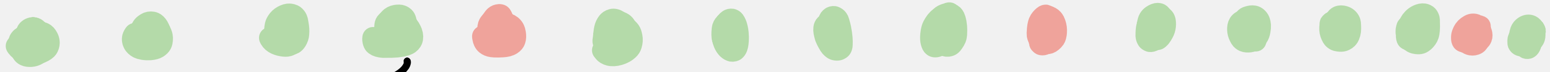
26 is even! So can't end up with 26!

Intro problems

KING'S MATHS SCHOOL

2. There are 16 pennies on a table. Two players take it in turns to each remove between 1 and 4 pennies (inclusive). The winner is the last person who can remove any pennies. Who has a winning strategy? [Hint: try with a smaller number of starting pennies]

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16.



you can win by taking **all** the coins.

whatever you take, you leave the opponent in a 'winnable' position.

Intro problems

KING'S MATHS SCHOOL

2. There are 16 pennies on a table. Two players take it in turns to each remove between 1 and 4 pennies (inclusive). The winner is the last person who can remove any pennies. Who has a winning strategy? [Hint: try with a smaller number of starting pennies]

you remove 1

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

you can win by taking all the coins.

whatever you take, you leave the opponent in a winnable position.

remove 2

remove 3

remove 4

Remove coins so as to leave a multiple of 5 remaining!

Combinatorial games

KING'S MATHS SCHOOL

Very loosely, combinatorial games have the following properties:

- Two players, who alternate turns and both know exactly what is going on – "complete information", with no random/hidden element, and will both try and win!
- Has several (usually a finite number of) positions, with a designated "starting position"
- A **move** consists of going from one position to another position
- The new positions that a player may choose to are called the **options** for that position
- Can be an **impartial game**: both players have the same options in any given position
- Or can be a **partisan game**: each player has different options in any given position
- Game must always end with a clear winner (no draws!).

Pennies game from before: **impartial** game.

Chess: not combinatorial, each player has different moves in a given position (and the game could also end in a draw)!

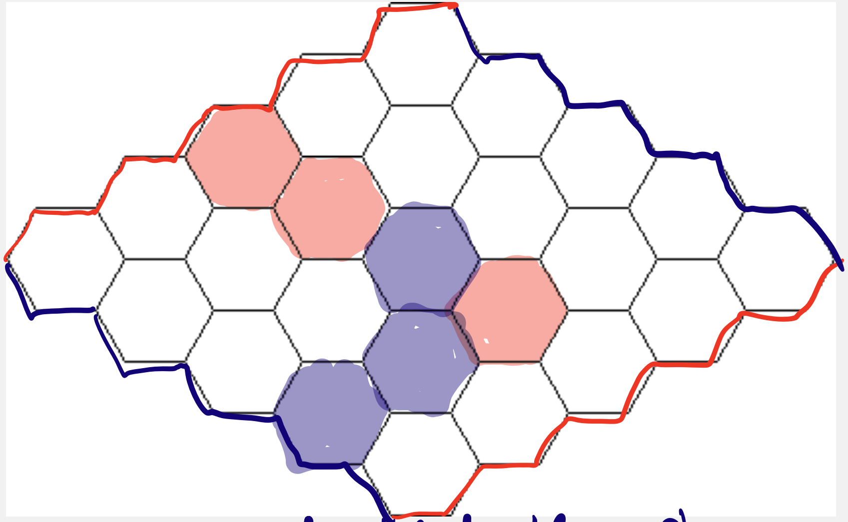
Bridge/poker: not combinatorial, incomplete information. → hidden element.

Hex → as seen on the TV Gameshow "Blockbusters"

Two players, take it in turns to put one of their coloured counters on the board.

Objective: make a continuous path/bridge between their two edges.

Partizan game (because both players have different colours)!



It turns out that the first player can force a win - but: we don't know what the winning strategy is!

Analysing combinatorial games

KING'S MATHS SCHOOL

Normally, we want to analyse the game to see if we can come up with a winning strategy for either player.

Ideas seen so far:

- Look at **parity** (even/oddness of certain quantities)
- **Symmetry/strategy-stealing** (do the same thing that your opponent just did, or do something to "complement" what they did).

Some other ideas:

- Draw a "tree" to show the possibilities at each stage, splitting into branches
- Be cleverer in how you can display "winning/losing" starting positions…

Problems #1

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

1. There are 40 coins on the table. Two players alternate, taking between 1 and 10 coins (inclusive) on each turn. The player who takes the last coin wins the game (or who leaves too few coins for the opponent to take).

2. Similar to 1. 500 stones on the table. Each player can take between 1 and k stones (inclusive) on each turn. The player who takes the last stone wins the game. [Describe how to figure out the winner for a general k!]

3. There are two heaps of coins on the table, one with 100 coins and one with 200. Two players alternate, taking as many coins as they wish from just one of the piles. The player who takes the last coin wins the game.

4. Start with a positive integer – say, 13. Players alternate subtracting a positive square number from this (but are not allowed to go into the negatives). The player who leaves the number 0 wins the game.

5. Start with a collection of 3 cats and 5 dogs. Players alternate as follows: a legal move is removing any number of cats, or any number of dogs, or an equal number of cats and dogs (but always at least one). If a player cannot do this, they lose.

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

1. There are 40 coins on the table. Two players alternate, taking between 1 and 10 coins (inclusive) on each turn. The player who takes the last coin wins the game.

First player takes 7, to leave 33.

Thereafter the first player responds to the second player, to ensure there are a multiple of 11 coins left.

$(1+10=11)$

"If player 2 takes n, player 1 will take $11-n$".

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

2. Similar to 1. 500 stones on the table. Each player can take between 1 and k stones (inclusive) on each turn. The player who takes the last stone wins the game. [Describe how to figure out the winner for a general k!]

Strategy: leave a multiple of $k+1$ coins for the other player.

But: player 2 can force a win when $k+1$ is a factor of 500.

$\Rightarrow k = 1, 3, 4, 9, 19, 24, 49, 99, 124, 249, 499.$

For all other values of k, player 1 can force a win.

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

3. There are two heaps of coins on the table, one with 100 coins and one with 200. Two players alternate, taking as many coins as they wish from just one of the piles. The player who takes the last coin wins the game.

Player 1: take 100 coins from the 200 pile, leaving (100, 100).

Then: mirror what player 2 does, but to the other pile.

eg. (100, 100) → (90, 100) → (90, 90) → (90, 70) → (70, 70) → . . . .

This ensures player 1 can always win!

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

4. Start with a positive integer – say, 13. Players alternate subtracting a positive square number from this (but are not allowed to go into the negatives). The player who leaves the number 0 wins the game.

1 2 3 4 5 6 7 8 9 10 11 12 13

Have to remove 1, leaves the other player in a winning position.

"Square numbers are winning – choose to subtract that square".

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

4. Start with a positive integer – say, 13. Players alternate subtracting a positive square number from this (but are not allowed to go into the negatives). The player who leaves the number 0 wins the game.

1 2 3 4 5 6 7 8 9 10 11 12 13

?

Have to remove 1, leaves the other player in a winning position.

Remove 1, leaves the other player in a losing position.

Q.

Problems #1: rough solutions

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

4. Start with a positive integer – say, 13. Players alternate subtracting a positive square number from this (but are not allowed to go into the negatives). The player who leaves the number 0 wins the game.

1 2 3 4 5 6 7 8 9 10 11 12 13

?

Have to remove 1, leaves the other player in a winning position.

Remove 1, leaves the other player in a losing position.

No matter what you pick, you leave the opponent in a winning position.

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

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1 2 3 4 5 6 7 8 9 10 11 12 13

?

Have to remove 1, leaves the other player in a winning position.

Remove 1, leaves the other player in a <u>losing</u> position.

No matter what you pick, you leave the opponent in a winning position.

Repeat previous arguments to fill in 6, 7, 8

q.

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

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1 2 3 4 5 6 7 8 9 10 11 12 13

?

Have to remove 1, leaves the other player in a winning position.

Remove 1, leaves the other player in a losing position.

No matter what you pick, you leave the opponent in a winning position.

Repeat previous arguments to fill in 10, 11, 12

Problems #1: rough solutions

KING'S MATHS SCHOOL

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

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1 2 3 4 5 6 7 8 9 10 11 12 13

?

Have to remove 1, leaves the other player in a winning position.

Remove 1, leaves the other player in a losing position.

No matter what you pick, you leave the opponent in a winning position.

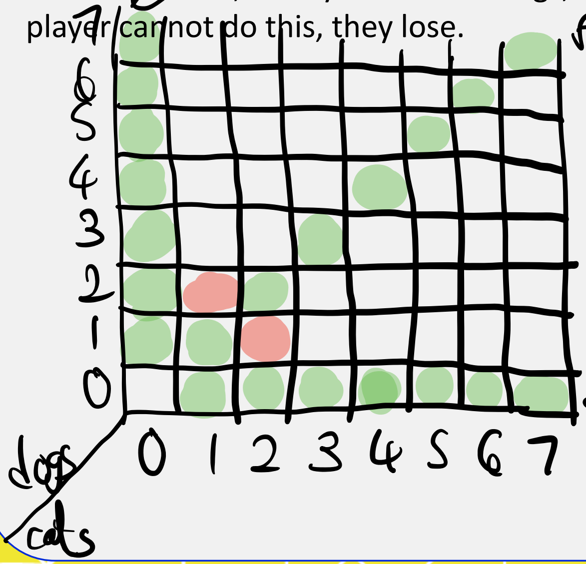
Must pick 1 to guarantee a win!

Problems #1

remove all dogs.

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

5. Start with a collection of 3 cats and 5 dogs. Players alternate as follows: a legal move is removing any number of cats, or any number of dogs, or an equal number of cats and dogs (but always at least one). If a player cannot do this, they lose.



remove equal no. of cats and dogs

remove all cats

= winning positions

= losing position, because no matter what you do, you leave your opponent in a <u>winning</u> position

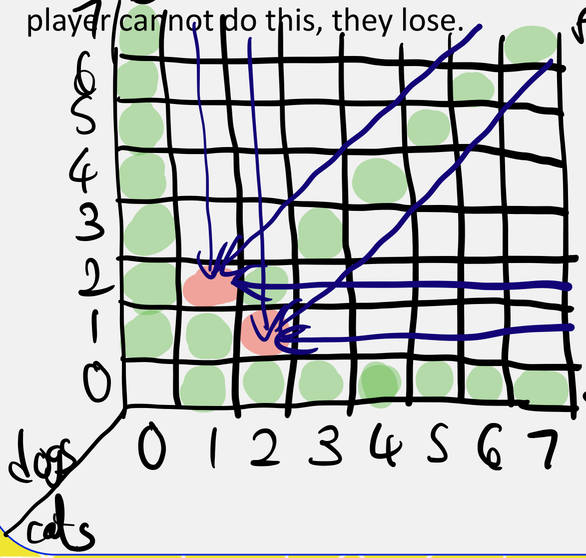
Problems #1

KING'S MATHS SCHOOL

remove all dogs.

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

5. Start with a collection of 3 cats and 5 dogs. Players alternate as follows: a legal move is removing any number of cats, or any number of dogs, or an equal number of cats and dogs (but always at least one). If a player cannot do this, they lose.



remove equal no. of cats and dogs

remove all cats

= winning positions

= losing position.

If you can leave your opponent at (1,2) or (2,1) then this is winning for you!

Problems #1

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

5. Start with a collection of 3 cats and 5 dogs. Players alternate as follows: a legal move is removing any number of cats, or any number of dogs, or an equal number of cats and dogs (but always at least one). If a player cannot do this, they lose.

So, can fill in more green spots (places where you can then leave your opponent in a losing position).

Problems #1

In each case: who has a winning strategy, and what is it? It might help to change the yellow-highlighted numbers to look for patterns/explore/make your own conjectures!

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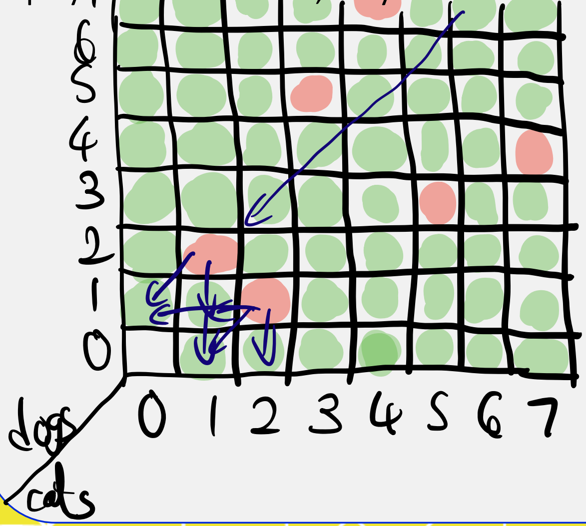


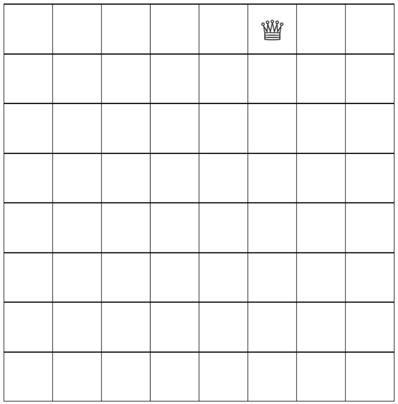
From (1,2) or (2,1), no matter what you do, you leave your opponent in a winning position!

So (3,5) and (5,3) are losing. So player 2 can force a win

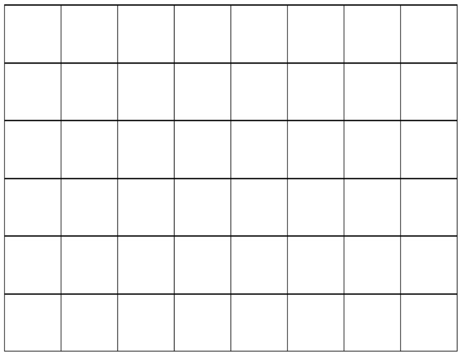
Next, (4,7) and (7,4) are also losing

Problems #2 (part I)

1. A queen is on a chessboard like in the picture on the right.
Two players take it in turns to move it, but can only move it closer to the bottom-left corner than when they started. So, they can only move it left, down or diagonally down and left any number of squares. The player who is compelled to move it into the bottom-left corner *loses*.
Who has a winning strategy in this case?



2. Start with a rectangular chocolate bar which is 6×8 squares in size. Two players take it in turns to break a piece of chocolate along a single straight line bounded by the squares. For example, you could break the original chocolate into a 6×2 and a 6×6 piece. The other person could then break this latter piece into a 1×6 and a 5×6 piece. The player who cannot break any of the chocolate further (i.e. when everything remaining is a 1×1 piece) loses.
Who wins in this case? What about other size rectangles?



Problems #2 (part II)

KING'S MATHS SCHOOL

3. There are 20 stones on the table. Each player can take 1 or a prime number of stones. The player who takes the last stone wins the game.

4. There are 50 stones on the table. Each player can take 1 or a prime number or a prime power number of stones. The player who takes the last stone wins the game.

Hint for q3 and q4: try to spot similarities in the strategy, compared to the "you can take 1, 2, … , n stones" version we looked at earlier.

5. There are n $(n \geq 3)$ pennies on the table, initially forming a single stack. On each turn, a player chooses one of the stacks (which has at least 3 coins in it), and splits it into two smaller stacks (it is up to them exactly how they distribute the coins across these two smaller piles). When a player makes a move which causes all the stacks to only have 1 or 2 coins in them, they win.

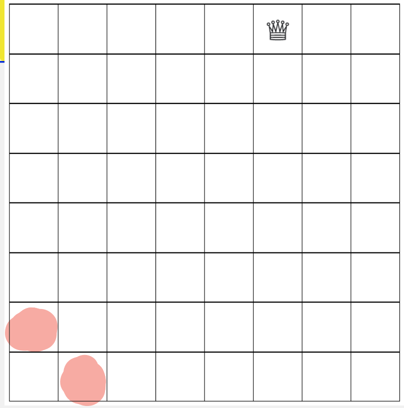
For different values of n, who has a winning strategy, and what is it?

Hint: look at even values of n first, then look at small values of n, then make a conjecture and try to prove it!

Problems #2: rough solutions

KING'S MATHS SCHOOL

1. A queen is on a chessboard like in the picture on the right.
Two players take it in turns to move it, but can only move it closer to the bottom-left corner than when they started. So, they can only move it left, down or diagonally down and left any number of squares. The player who is compelled to move it into the bottom-left corner *loses*.
Who has a winning strategy in this case?

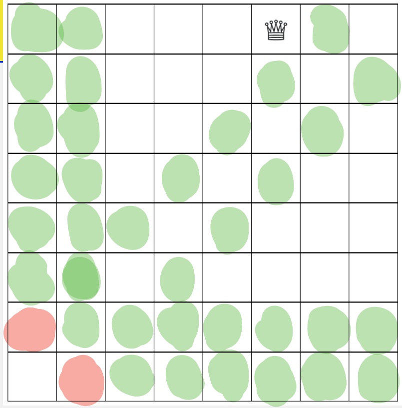


(1,0) and (0,1) are losing squares.

Problems #2: rough solutions

KING'S MATHS SCHOOL

1. A queen is on a chessboard like in the picture on the right.
Two players take it in turns to move it, but can only move it closer to the bottom-left corner than when they started. So, they can only move it left, down or diagonally down and left any number of squares. The player who is compelled to move it into the bottom-left corner *loses*.
Who has a winning strategy in this case?

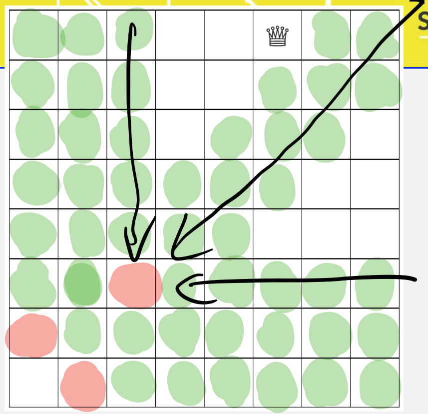


(1,0) and (0,1) are losing squares, so if we can put the opponent there, then they lose (i.e. we win).

Problems #2: rough solutions

KING'S MATHS SCHOOL

1. A queen is on a chessboard like in the picture on the right.
Two players take it in turns to move it, but can only move it closer to the bottom-left corner than when they started. So, they can only move it left, down or diagonally down and left any number of squares. The player who is compelled to move it into the bottom-left corner *loses*.
Who has a winning strategy in this case?



(2, 2) is losing.

Problems #2: rough solutions

KING'S MATHS SCHOOL

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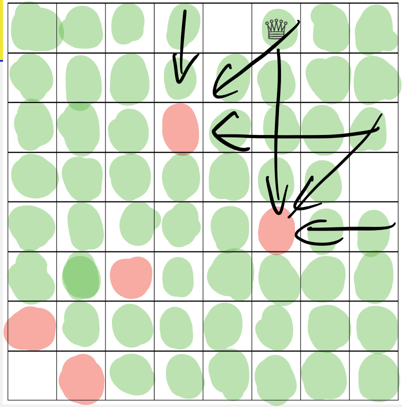
Who has a winning strategy in this case?



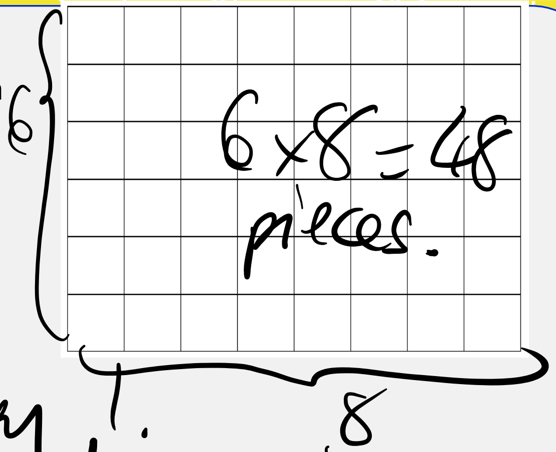
(4,6) and (6,4) are losing. So, player 1 has a winning starting position, but only with correct play.

Leave the opponent on a red square after your move.

Problems #2: rough solutions

KING'S MATHS SCHOOL

2. Start with a rectangular chocolate bar which is 6×8 squares in size. Two players take it in turns to break a piece of chocolate along a single straight line bounded by the squares. For example, you could break the original chocolate into a 6×2 and a 6×6 piece. The other person could then break this latter piece into a 1×6 and a 5×6 piece. The player who cannot break any of the chocolate further (i.e. when everything remaining is a 1×1 piece) loses. Who wins in this case? What about other size rectangles?



Note: total no. of pieces increases by 1.

After player 1 has moved: even number of pieces.

After player 2 has moved: odd number of pieces.

48 is **even**, so player 1 always wins.

Problems #2: rough solutions

KING'S MATHS SCHOOL

3. There are 20 stones on the table. Each player can take 1 or a prime number of stones. The player who takes the last stone wins the game. → note: can certainly take 1, 2, or 3 stones (but not 4)

So this is like the earlier game with $k=3$.

Player 2 can win: if player 1 removes n stones, player 2 responds by taking $4-(n \bmod 4)$ stones.

4. There are 50 stones on the table. Each player can take 1 or a prime number or a prime power number of stones. The player who takes the last stone wins the game.

Note: can certainly take 1, 2, 3, 4, 5 stones (but not 6).

So this is like the earlier game with $k=5$.

Player 1 can win: remove 2 stones.

Then, if player 2 removes n stones, player 2 responds by taking $6-(n \bmod 6)$ stones.

Problems #2: rough solutions

KING'S MATHS SCHOOL

5. There are n ($n \geq 3$) pennies on the table, initially forming a single stack. On each turn, a player chooses one of the stacks (which has at least 3 coins in it), and splits it into two smaller stacks (it is up to them exactly how they distribute the coins across these two smaller piles). When a player makes a move which causes all the stacks to only have 1 or 2 coins in them, they win.

For different values of n, who has a winning strategy, and what is it?

Hint: look at even values of n first, then look at small values of n, then make a conjecture and try to prove it!

If n is even: first split into two piles of $\frac{n}{2}$ stones. Then whatever player 2 does to one of these piles, player 1 should "mirror" it on the other pile!

$\Rightarrow$ Player 1 can win.

Problems #2: rough solutions

KING'S MATHS SCHOOL

3 4 5 6 7 8 9 10 11 12 ...

?

If $n=3$, player 1 splits it into $1+2$ and therefore automatically wins

Problems #2: rough solutions

KING'S MATHS SCHOOL

3 4 5 6 7 8 9 10 11 12 ...

No matter how player 1 splits (2+3 or 1+4), player 2 is left to manipulate the pile of 3 or 4, so player 2 wins.

Problems #2: rough solutions

3 4 5 6 7 8 9 10 11 12 . . .

For all other odd numbers, split into 3 + (even).
Player 1 can win as follows:

→ respond to player 2 by working on the pile they worked on

→ at some stage, player 2 will "win" one pile...

→ ... leaving player 1 to be "first" on the other pile, and so pl 1 will "win" that!

Nim

KING'S MATHS SCHOOL

Setup: there are two or more piles of stones.
Each turn, a player can remove as many stones as they want (but at least one) from one of the piles.

The player who takes the last stone wins.

GOAL: who has a winning strategy, and what is it?

Nim problems

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

1. Two piles, 2 and 3 stones.
2. Two piles, 3 and 3 stones
3. Two piles, different number of stones in each.
4. Two piles, same number of stones in each.

5. Three piles, and you know that there are two piles which contain the same number of stones.
6. Three piles, of 1, 2 and 3 stones.
7. Three piles, of 1, 2 and n stones ($n \geq 4$).

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

1. Two piles, 2 and 3 stones.

3. Two piles, different number of stones in each.

Player 1 can always win

1st move: "equalise" the two piles.

Subsequent moves: "mirror" what player 2 does, but on the <u>other</u> pile.

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

2. Two piles, 3 and 3 stones

4. Two piles, same number of stones in each.

Player 2: they can "mirror" all of player 1's moves.

5. Three piles, and you know that there are two piles which contain the same number of stones.

Player 1: they take all in the 3rd pile (to leave two piles with the same number of stones).
This is <u>losing</u> by 2+4, for player 2.

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

6. Three piles, of 1, 2 and 3 stones.

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

6. Three piles, of 1, 2 and 3 stones.

Player 1 must choose which pile (1, 2, 3) to remove stones from.

Pile of 1 → this will leave two unequal piles (2, 3) for player 2, which is winning for player 2 by q3. So player 1 loses.

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

6. Three piles, of 1, 2 and 3 stones.

<u>Pile of 2</u> → leaves (1, 3) or (1, 1, 2) for player 2.
By q3 and q5 these are winning for player 2, so losing for player 1.

<u>Pile of 3</u> → leaves (1, 2, 2) or (1, 2, 1) or (1, 2).
By q5, q5 and q3, these are winning for player 2 so losing for player 1.

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

6. Three piles, of 1, 2 and 3 stones.

Conclusion: no matter what player 1 does, player 2 is left in a position from which they can guarantee the win.

So, player 2 has a winning strategy here.

Nim problems: rough solutions

KING'S MATHS SCHOOL

Which player has a winning strategy – the one going first or second?

7. Three piles, of 1, 2 and n stones ($n \geq 3$).

If $n=3$, player 2 has a winning strategy by q6.

If $n \geq 4$, player 1 should remove $n-3$ stones from the pile of n. This leaves the piles as $(1,2,3)$ for player 2, which is a losing position for player 2.

So for $n \geq 4$, player 1 has a winning strategy!

Nim: general strategy and solution

KING'S MATHS SCHOOL

Nim is a more difficult game to analyse by just drawing decision trees and arguing logically, because of how many different options there are.

The winning strategy involves computing something called the "Nim sum" of all the piles:

- Convert the size of each pile to a binary number
- XOR all of these binary strings together (this is called the "Nim sum")
- XOR the Nim sum with each of the original pile numbers
- (At least) one of these results will be less than the corresponding pile number: pick that pile and take as many stones to reduce it to that result.

See the Wikipedia article on Nim for examples and a proof.

Further (and some more difficult) problems

KING'S MATHS SCHOOL

1. 20 dominoes are standing in a row vertically. Two players A and B take it in turn to remove either one or two consecutive dominoes (note that if a gap forms in the row, the two dominoes either side of the gap are *not* considered consecutive, and cannot be removed in the same turn). The player who takes the last domino wins. Who has a winning strategy, and what is it?

2. Consider the following modified version of the game: instead of the dominoes standing in a row, they are standing in a circle. The player who takes the last domino wins.
Who has a winning strategy, and what is it?

3. Starting from 1, two players take it in turns multiplying the current number by any whole number from 2 to 9 inclusive. The player who first names a number greater than 100 wins.
Which player, if either, can guarantee victory?
What if the aim is to name a number greater than 1000?

4. There are two piles of sweets, one with 20 and the other with 21. Two players take turns eating all the sweets in one pile, and then separating the other into two (not necessarily equal) piles. The player who cannot eat any sweets on their turn loses. Which player, if either, can guarantee victory and what is their strategy?

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5. The number 60 is written on a blackboard. Two players take it in turns to subtract one of the divisors from the number, and replace the number with the result of the subtraction. The payer who writes the number 0 loses.

6. A box contains 300 matches. Players take turns removing no more than half of the matches in the box. The player who cannot remove any matches loses.

7. On the TV show "Fortune of Mathematics", a contest is held among several players. Each player initially has a heap of 100 stones; the player divides the heap into two parts, then divides one of the parts into two again, etcetera, until the player has 100 separate stones. After each division, the player records the product of the numbers of stones in the two new heaps, and at the end the player adds up all of these products.

The player with the largest final sum N wins the game and receives an award of £$N \times 201$. The players can see each others moves *after* a move has been made, and the TV host makes sure that everyone completes their next move BEFORE letting everyone see each other's positions.

Is there a strategy for one of the players to become a millionaire? Note that draws of highest sums do NOT win anyone any money.

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(Bay Area Mathematical Olympiad 2002 Q3)
8. There are n ($n \geq 3$) pennies on the table, initially forming a single stack. On each turn, a player chooses one of the stacks, and splits it into two smaller stacks (it is up to them exactly how they distribute the coins across these two smaller piles). When a player makes a move which causes all the stacks to only have 1 or 2 coins in them, they win.

For different values of n, who has a winning strategy, and what is it?

Hint: look at small values of n (up to about 12 should be sufficient), then make a conjecture and try to prove it! You should focus on proving the cases where the first player can win, first. The other case is trickier to argue.