Starter Questions

KING'S MATHS SCHOOL

1) Suppose you have a pizza. If you slice the pizza with one straight cut, it divides the pizza into two pieces. You can cut the pizza any way you like.
   If you cut through the pizza again how many pieces do you get?
   What about for 3, 4 or 5 cuts?

   Can you work out what the maximum numbers of pieces you will have for n cuts?

2) What's the next two numbers in the following sequences? Can you work out the nth terms?

   a) 2, 5, 10, 17, ___, ___, …

   b) 1, 3, 7, 15, ___, ___, ….

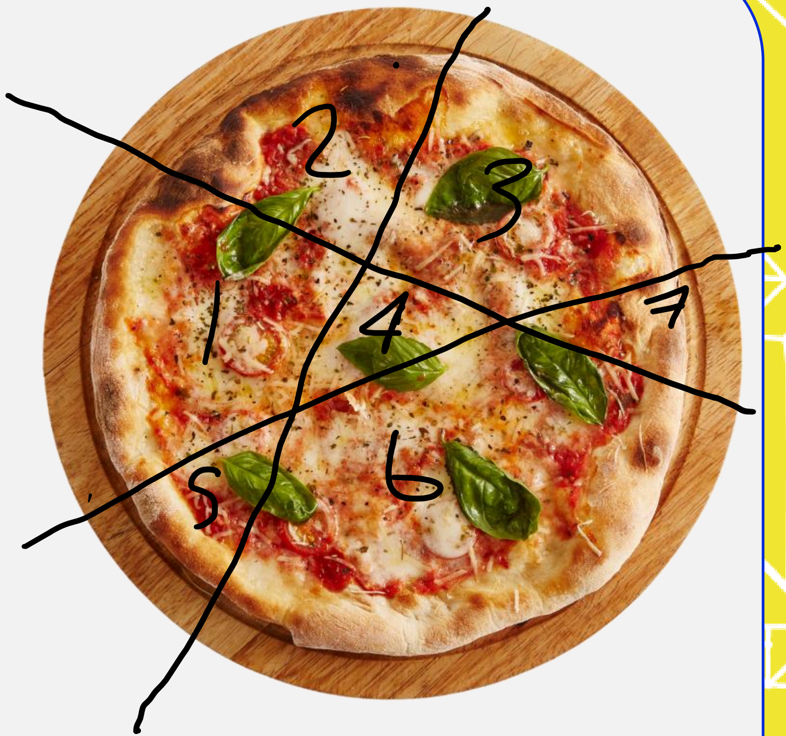
3) What is 1+2+….+99+100? Can you guess a formula for the sum of all integers from 1 to n?

Starter Questions

1) Suppose you have a pizza. If you slice the pizza with one straight cut, it divides the pizza into two pieces. If you cut through the pizza again how many pieces do you get? What about for 3, 4 or 5 cuts?

Can you work out what the maximum numbers of pieces you will have for n cuts?

n	pieces
1	2
2	4
3	7
4	11
⋮	⋮
n	$f(n)$



Starter Questions

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2) What's the next two numbers in the following sequences? Can you work out the nth terms?

a) 2, 5, 10, 17, __, __, ...

n^2+1

$2, 5, 10, 17, 26, 37 \ldots$

$+3 \quad +5 \quad +7 \quad +9$

b) 1, 3, 7, 15, __, __, ...

$2 \quad 4 \quad 8$

$2^n - 1$

For $n=3$ $f(3) = 3^2+1 = 10$

$f(n) = n^2+1 \rightarrow$ nth term : Explicitly

$x_n = x_{n-1} + 2n - 1$ Recursive

$x_3 = x_2 + 2(3) - 1 = x_2 + 5 = 5+5 = 10$

Starter Questions

3) What is 1+2+….+99+100?   Can you guess a formula for the sum of all integers from 1 to 100?

Different approaches e.g.

Try some examples for smaller values to see if there's a pattern.

$1 = 1$

$1+2 = 3$

$1+2+3 = 6$

$1+2+3+4 = 10$

$\vdots$

$1+2+\dots+100 =$

$1+2+\dots\dots+99+100$

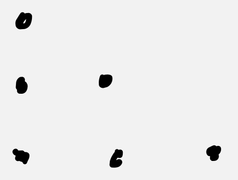
101

101

50 pairs of 101

$50 \times 101 = 5050$

Geometric example



Notation Recap

For today's session the following notation may be helpful:

Summation: $1+2+\ldots+99+100 = \sum_{k=1}^{100} k$

$\sum_{k=1}^{n} k = 1+\ldots+n$

Types of sequences:

Explicit sequence: An explicit definition of a sequence is where we write the terms in the form $x_k = f(k)$ where f is a function.

Recursive sequence: A recursive definition of a sequence is where we write the next term of a sequence in terms of previous terms. E.g. $x_k = f(x_{k-1}, x_{k-2}, \ldots)$. A value is given for the first few terms.

Induction

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Mathematical induction is a method of proof within mathematics.

Two key steps:

1) Show the statement is true for an initial case e.g. n=0 or n=1. } Base case
2) Assume the statement is true for $n = \mathrm{k}$.

Then show if the statement is true for $n = k$, then it also holds for $n = k + 1$. } Inductive step

$$x_0 \to x_1 \to x_2 \to \ldots - - -$$

Property $P(n)$ = property of nth term

Check: $P(0)$ ✓ $P(1)$ ✓ for example

Assume $P(k)$ is true $\to$ Prove that $P(k+1)$

Induction-Examples of Inductive Step

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a) Use induction to verify that $\sum_1^n k = \frac{n(n+1)}{2}$

b) Verify that the maximum number of pizza pieces after n cuts is $\frac{n(n+1)}{2} + 1$

Induction-Examples of Inductive Step

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a) Use induction to verify that $\sum_1^n k = \frac{n(n+1)}{2}$

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a)

n	$\sum^n k$	$\frac{n(n+1)}{2}$	
1	1	$\frac{1(1+1)}{2} = 1$	✓ Base case
2	$1+2 = 3$	$\frac{2(2+1)}{2} = 3$	
3	$1+2+3=6$	$\frac{3(3+1)}{2} = 6$	
⋮			
K	$\sum_1^K k =$	$\frac{K(K+1)}{2}$	Assume true

Showed formula holds for base case

Assumed $\sum_1^K k = \frac{K(K+1)}{2}$ is true

Now prove formula holds for $n = K+1$.

What is $\sum_1^{K+1} k$?

Induction-Examples of Inductive Step

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a) Use induction to verify that $\sum_1^n k = \frac{n(n+1)}{2}$

b) Verify that the maximum number of pizza pieces after n cuts is $\frac{n(n+1)}{2} + 1$

a)

What is $\sum_1^{k+1} k$?

$$\sum_1^{k+1} k = \underbrace{\sum_1^{k} k}_{} + \underbrace{(k+1)}_{}$$

inductive step

$$= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\therefore \sum_1^{k+1} k = \frac{(k+1)(k+2)}{2} \rightarrow \frac{n(n+1)}{2} \text{ when } n = k+1.$$

Induction-Examples of Inductive Step

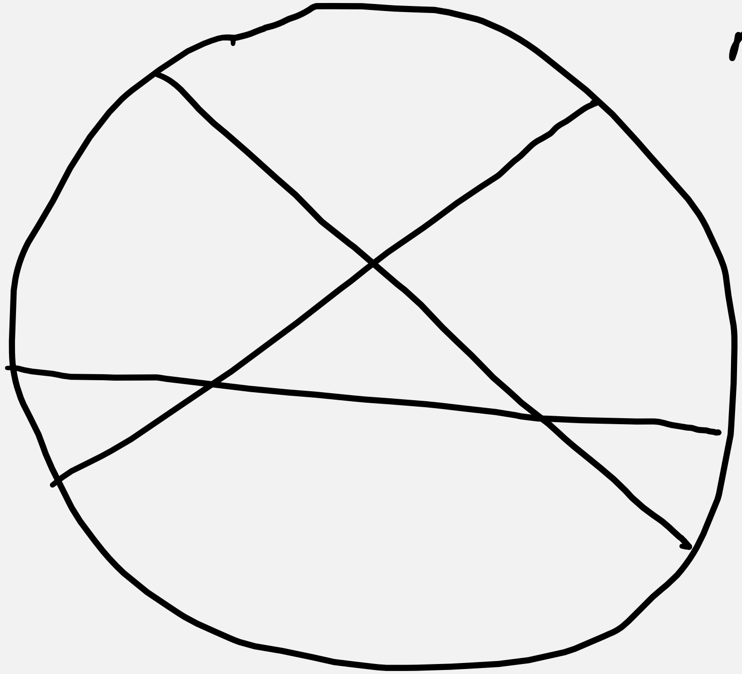
KING'S MATHS SCHOOL

a) Use induction to verify that $\sum_1^n k = \frac{n(n+1)}{2}$

b) Verify that the maximum number of pizza pieces after n cuts is $\frac{n(n+1)}{2} + 1$

} Similar formula.

b) What is the inductive step?



nth cut adds n pieces

$P(n)$ = no. of pieces after n cuts.

$P(0) = \frac{0(0+1)}{2} + 1 = 1$ ✓ (whole pizza)

$P(1) = \frac{1(1+1)}{2} + 1 = 2$

Assume true for $P(k)$

Then prove for $P(k+1)$

$k+1$

Induction Questions

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1) Using induction prove the general formula for the following starter sequences:

    a) 2, 5, 10, 17, ___, ___, … General formula: $a_n = n^2 + 1$

    b) 1, 3, 7, 15, ___, ___, …. General formula: $b_n = 2^n - 1$

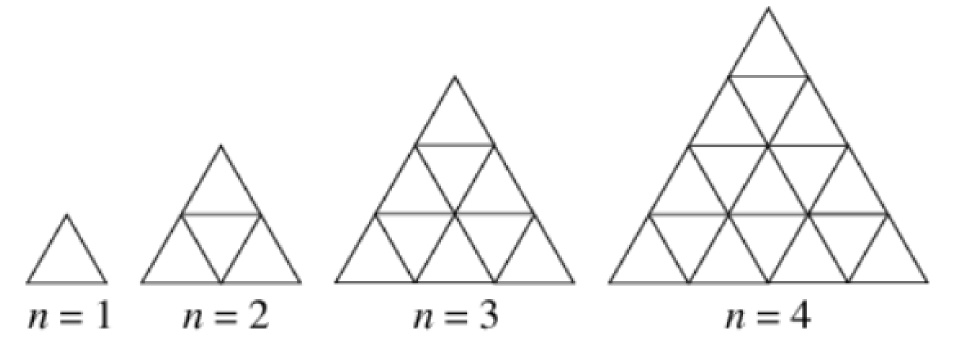
2) Prove that the sum of the first n odd numbers is n^2.

3) Define the following sequence c_n of numbers as follows: $c_0 = 2, c_1 = 3$ and then $c_n = 3c_{n-1} - 2c_{n-2}$ for n > 2. Guess a general formula for c_n and verify it using induction.

4) Use induction to prove that $\sum_1^n k^2 = \frac{1}{6}n(n+1)(2n+1)$

5) Guess a general formula for the sum $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{(n-1)n}$ where n > 2. Prove your formula is correct by induction.

6) Look at the sequence of shapes on the right.
Find a formula for the number of identical small triangles in the nth shape. Can you prove your formula using induction?



Induction Questions

KING'S MATHS SCHOOL

1) Using induction prove the general formula for the following starter sequences:

a) 2, 5, 10, 17, ___, ___, … General formula: $a_n = n^2 + 1$

Show/Prove $a_{k+1} = (k+1)^2 + 1$

b) 1, 3, 7, 15, ___, ___, …. General formula: $b_n = 2^n - 1$

Base case e.g. a_1

Inductive

$\underbrace{a_k}_{\text{assumption}}$ ✓ $\Rightarrow$ a_{k+1} prove

2) Prove that the sum of the first n odd numbers is n^2.

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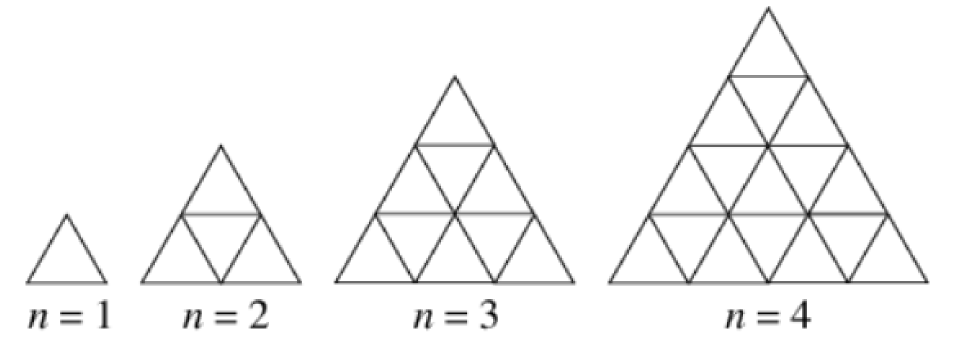

Induction Questions

KING'S MATHS SCHOOL

1) Using induction prove the general formula for the following starter sequences:
   a) 2, 5, 10, 17, __, __, ...   General formula: $a_n = n^2 + 1$

$$\boxed{a_{n+1} = a_n + 2n + 1}$$

2, 5, 10, 17, 26, 37

+3 +5 +7 +9 +11

n	a_n	n^2+1
1	2	$1^2+1=2$ ✓ Base case : $n=1$
2	5	$2^2+1=5$
⋮		
k	a_k	k^2+1 → Assume true

$$a_{k+1} = a_k + 2k + 1$$
$$= (k^2+1) + (2k+1)$$
$$= (k^2+2k+1) + 1$$
$$= (k+1)^2 + 1$$

So $a_{k+1} = (k+1)^2 + 1$

This is formula for a_n with $n = k+1$

Induction Questions

KING'S MATHS SCHOOL

3) Define the following sequence b_n of numbers as follows: $c_0 = 2, c_1 = 3$ and then $\boxed{c_n = 3c_{n-1} - 2c_{n-2}}$ for n > 2. Guess a general formula for c_n and verify it using induction.

$c_0 \quad c_1 \quad c_2 \quad c_3 \ldots c_4$

$\underbrace{2 \quad 3}_{+1} \underbrace{\quad 5}_{+2} \underbrace{\quad 9}_{+4} \underbrace{\ldots 17}_{+8}$ → powers of 2 each time.

$2^0 = 1 \quad 2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 8$

$c_n = 2^n + 1$

→ $n=0$ $\quad c_0 = 2^0 + 1 = 2$ ✓

→ $n=1$ $\quad c_1 = 2^1 + 1 = 3$ ✓

$c_2 = 3c_1 - 2c_0$
$= 3(3) - 2(2) = 5$

$c_3 = 3c_2 - 2c_1$
$= 3(5) - 2(3) = 15 - 6$
$= 9$

Base case

Inductive step.

Induction Questions

KING'S MATHS SCHOOL

3) Define the following sequence b_n of numbers as follows: $c_0 = 2, c_1 = 3$ and then $\boxed{c_n = 3c_{n-1} - 2c_{n-2}}$ for n > 2. Guess a general formula for c_n and verify it using induction.

$\underline{C_n = 2^n + 1}$ → Shown for $n = 0$, $n = 1$. ✓

$\underline{C_K = 2^K + 1}$ → Assume true for $n = K$ ✓

↳ Prove true for $\underline{n = K+1}$: Prove $\underline{C_{K+1} = 2^{(K+1)} + 1}$

$$C_{K+1} = 3C_K - 2C_{K-1} = 3(2^K + 1) - \underline{2(2^{K-1} + 1)}$$

$$= 3 \times 2^K + 3 - 2^K - 2 = 3 \times 2^K - 2^K + 1 = 2 \times 2^K + 1 = 2^{K+1} + 1$$

$n = 0$, $n = 1$ ✓ $n = K$ → show $n = K+1$

Induction Questions-Extension

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7) Prove that $a^2 - 1$is divisible by 8 for all odd integers a.

8) Prove that $n^3 + 2n$ is divisible by 3 for all integers n

9) Prove that, if $x > 0$ is any fixed real number, then $(1 + x)^n > 1 + nx$ for all n ≥ 2

10) By writing the following products as an algebraic expression, use induction to show:

a) That the product of any 3 consecutive numbers is divisible by 6.

b) The product of 4 consecutive numbers is divisible by 24.

11) Prove that $3^n > n^2$ for n=1 and n=2. Use mathematical induction to prove that the inequality holds for all $n \geq 2$.

Fibonacci

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0, 1, 1, 2, 3, 5, 8, __, __

Recursive

$$F_0 = 0, \quad F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

Explicit

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Fibonacci

KING'S MATHS SCHOOL

Recursive

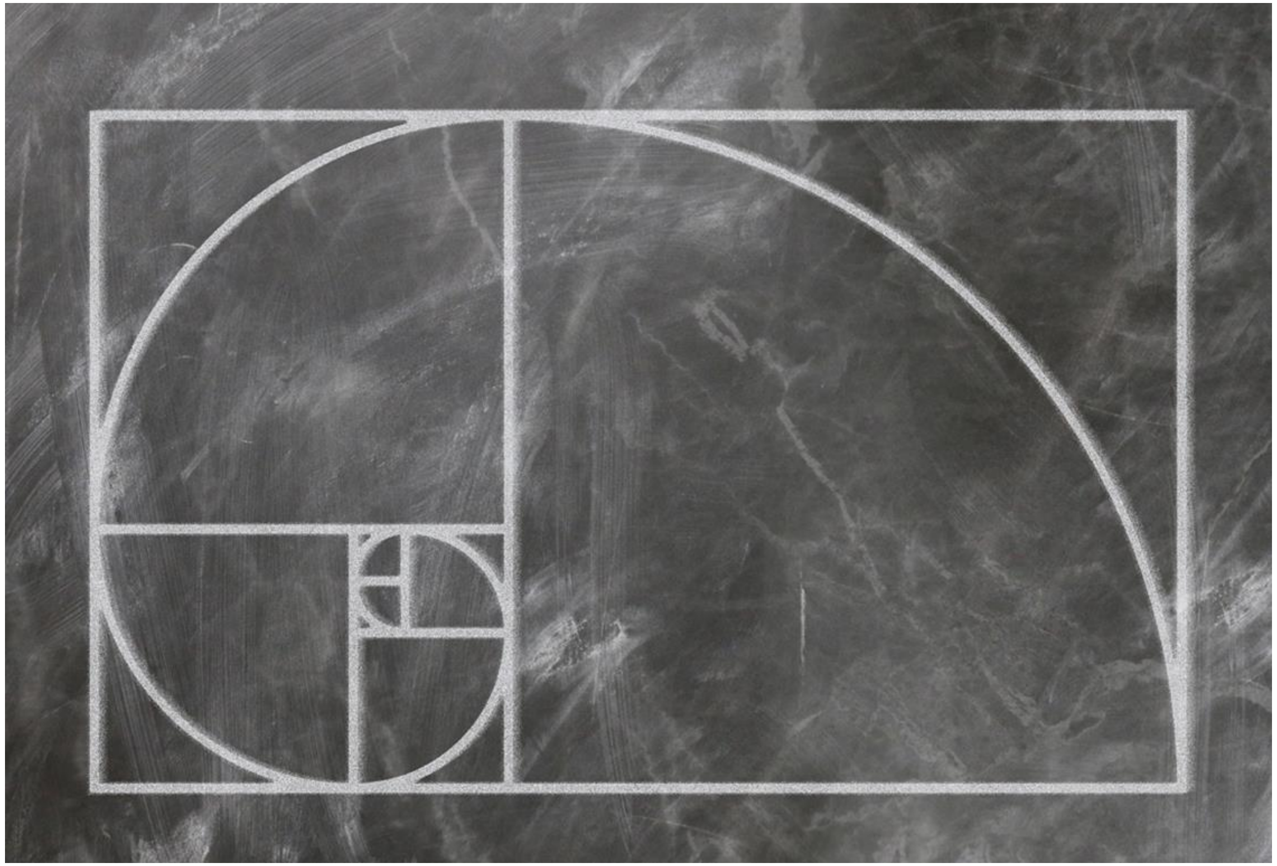
0, 1, 1, 2, 3, 5, 8,…

Explicit

$F_0 = 0, \quad F_1 = 1$

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$F_n = F_{n-1} + F_{n-2}$



Fibonacci

Recursive

$F_0 = 0, \quad F_1 = 1$

$F_n = F_{n-1} + F_{n-2}$

0, 1, 1, 2, 3, 5, 8,…

Explicit

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Extension Material

Resources:

Numberphile –Fibonacci ratio

https://www.youtube.com/watch?v=Nu-lW-Ifyec&t=320s

British Mathematical Olympiad Questions

Jan 1997 Q2

For positive integers n, the sequence $a_1, a_2, \ldots, a_n, \ldots$ is defined by $a_1 = 1$ and

$$a_n = \left(\frac{n+1}{n-1}\right)(a_1 + a_2 + \ldots + a_{n-1})$$

for $n \geqslant 2$. Determine the value of a_{1997}.

Jan 1996 Q2

A function f is defined over the set of all positive integers and satisfies $f(1) = 1996$ as well as

$$f(1) + f(2) + \ldots + f(n) = n^2 f(n)$$

for all $n \geqslant 2$. Calculate the exact value of $f(1996)$.