Starter Questions:
Complete the square in each question to solve the given equation.

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1. $x^2 + 6x + 8 = 0$
2. $x^2 - 10x + 20 = 0$
3. $x^2 + 8x + 9 = 0$
4. $x = \frac{1}{x-6}$
5. $\frac{x-3}{4} = \frac{7}{x+1}$
6. $x^2 + 3x - 1 = 0$
7. $2x^2 + 8x + 1 = 0$
8. $x^4 - 12x^2 + 25 = 0$

$(x + \square)^2 + \square = 0$

try $(x+3)^2 + ?$

$x^2 + 6x + 9 + ?$

equation can be rewritten as

$(x+3)^2 - 1 = 0$

$(x+3)^2 = 1$

$x+3 = 1$ or $x+3 = -1$

$x = -2$ or $x = -4$

Extensions: our shared expectations

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I will treat all other participants, students and teachers alike, with respect and with compassion.

I will, both during these sessions and afterwards, treat all participants equally regardless of their background or identity.

I will not, either during these sessions or afterwards, bully, harass, intimidate or discriminate against any participant in these sessions.

I will not record or capture any video or images (e.g. screenshots) during these sessions.

I will follow all instructions given to me to the best that I can.

I will engage in these sessions with tenacity and resilience. I will always 'have a go'.

From the warm up:

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$$\frac{x-3}{4} = \frac{7}{x+1}$$

$\times 4(x+1)$ $\quad \times 4(x+1)$

$(x-3)(x+1) = 28$

$x^2 - 3x + x - 3 = 28$

$x^2 - 2x - 31 = 0$

$(x-1)^2 - 32 = 0$

$(x-1)^2 = 32$

$x - 1 = \pm\sqrt{32} \Rightarrow x = 1 \pm \sqrt{32} = 1 \pm 4\sqrt{2}$

$\sqrt{32} = \sqrt{16} \times \sqrt{2}$

$= 4\sqrt{2}$

$$x^2 + 3x - 1 = 0$$

$(x + \frac{3}{2})^2 - \frac{13}{4} = 0$

$(x + \frac{3}{2})^2 = \frac{13}{4}$

$x + \frac{3}{2} = \pm\sqrt{\frac{13}{4}} = \pm\frac{\sqrt{13}}{2}$

$x = -\frac{3}{2} \pm \frac{\sqrt{13}}{2} = \frac{-3 \pm \sqrt{13}}{2}$

aside

$(x+\frac{3}{2})^2 = x^2 + 3x + \frac{9}{4}$

From the warm up:

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$2x^2 + 8x + 1 = 0$ $\Rightarrow$ $x^2 + 4x + \frac{1}{2} = 0$

$2(x+2)^2 - 7 = 0$ $(x+2)^2 - \frac{7}{2} = 0$

$2(x+2)^2 = 7$ $x = -2 \pm \sqrt{\frac{7}{2}}$

$(x+2)^2 = \frac{7}{2}$

$x = -2 \pm \sqrt{\frac{7}{2}}$

DISGUISED QUADRATIC

$x^4 - 12x^2 + 25 = 0$

$(x^2)^2 - 12x^2 + 25 = 0$

$k^2 - 12k + 25 = 0$

$(k-6)^2 - 11 = 0$

$(k-6)^2 = 11$

$k - 6 = \pm\sqrt{11}$

$k = 6 \pm \sqrt{11}$

$x^2 = 6 \pm \sqrt{11}$

$x = \pm\sqrt{6 \pm \sqrt{11}}$

$\sqrt{6+\sqrt{11}}$, $\sqrt{6-\sqrt{11}}$

$-\sqrt{6+\sqrt{11}}$, $-\sqrt{6-\sqrt{11}}$

$x^4 - 12x^2 + 25 = \sqrt{0}$

$x^2 - \sqrt{12}x + 5 = 0$

NOT OK!

let $k = x^2$,

What about this one?

$$x^2 + 4x + 5 = 0$$

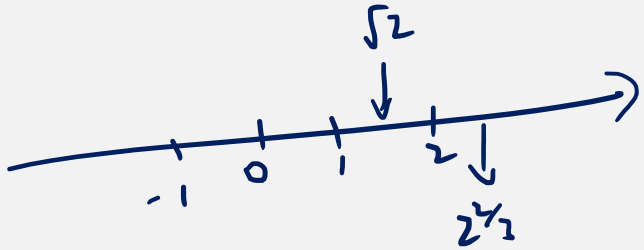
$(x+2)^2 + 1 = 0$

$(x+2)^2 = -1$ no "real solutions"

positive^2 = positive

negative^2 = positive

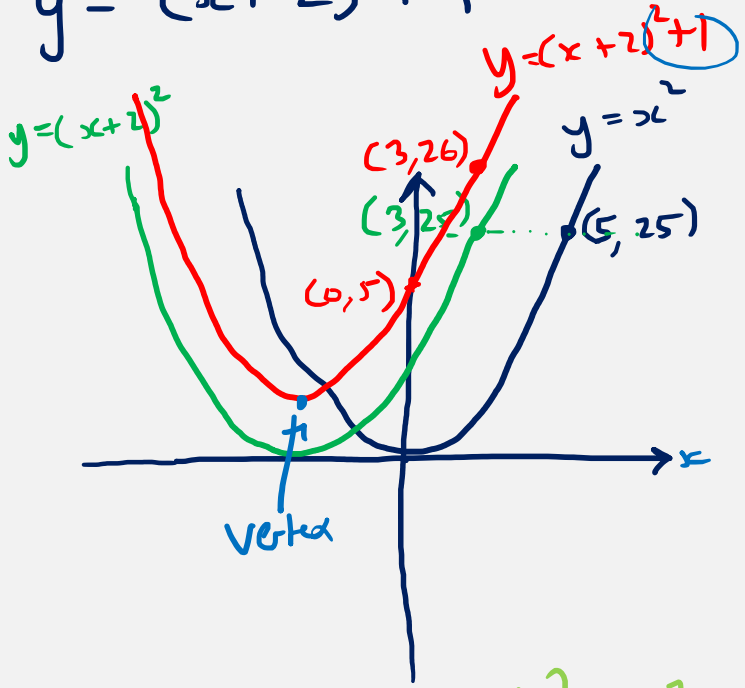
numbers on the number line



What's going on geometrically?

$y = x^2 + 4x + 5$

$y = (x+2)^2 + 1$



set $y = 0$

$x^2 + 4x + 5 = 0$

$(x+2)^2 + 1 = 0$

no solutions ∴ parabola is always above the y-axis

The smallest that $x^2+4x+5 = (x+2)^2+1$ can be is 1, this happens when $x = -2$.

When do you get what?

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For what values of k can does the equation $x^2-6x+k=0$ have solutions?

$x^2-6x+k=0$

$(x-3)^2-9+k=0$

$(x-3)^2=9-k$

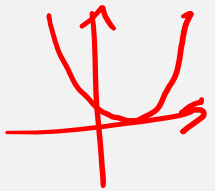
only proceed $9-k \geqslant 0$

$9 \geqslant k$

or $k \leqslant 9$.

$(x-3)^2=0$

$x=3$ ✓



Can we generalise to all quadratics?

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$$x^2 + px + q = 0$$

$$\left(x + \frac{p}{2}\right)^2 - \frac{p^2}{4} + q = 0$$

$$\left(x + \frac{p}{2}\right)^2 = \frac{p^2}{4} - q = \frac{p^2}{4} - \frac{4q}{4} = \frac{p^2 - 4q}{4}$$

$$\left(x + \frac{p}{2}\right) = \pm\sqrt{\frac{p^2 - 4q}{4}} = \pm\frac{\sqrt{p^2 - 4q}}{2}$$

$$x = -\frac{p}{2} \pm \frac{\sqrt{p^2 - 4q}}{2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Can we generalise to all quadratics?

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$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + px + q = 0$$

$$\Rightarrow x = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$\Rightarrow x = \frac{-\frac{b}{a} \pm \sqrt{\frac{b^2}{a^2} - \frac{4c}{a}}}{2}$$

$$= \frac{1}{2}\left[-\frac{b}{a} \pm \sqrt{\frac{b^2 - 4ac}{a^2}}\right] = \frac{1}{2}\left[-\frac{b}{a} \pm \frac{\sqrt{b^2 - 4ac}}{a}\right]$$

$$= \frac{1}{2}\left[\frac{-b \pm \sqrt{b^2 - 4ac}}{a}\right] = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

QUADRATIC FORMULA!

Minima, maxima

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Write $3x^2 + 9x + 10$ in the form $a(x + p)^2 + q$

What is the minimum value of the expression $3x^2 + 9x + 10$?

What value of x would you choose to make $3x^2 + 9x + 10$ as small as possible?

$$3x^2 + 9x + 10 = 3(x^2 + 3x) + 10$$

$$= 3\left(\left[x + \tfrac{3}{2}\right]^2 - \tfrac{9}{4}\right) + 10$$

$$= 3\left(x + \tfrac{3}{2}\right)^2 - \tfrac{27}{4} + 10$$

$$= 3\left(x + \tfrac{3}{2}\right)^2 + \tfrac{13}{4}$$

≥ 0

$a = 3,\ p = \tfrac{3}{2},\ q = \tfrac{13}{4}$

smallest output is $\tfrac{13}{4}$

get that when $x = -\tfrac{3}{2}$

Minima, maxima

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How large is it possible to make $-6 + 5x - x^2$ by choosing x appropriately?

Minima, maxima, and more!

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1. What are the minimum values of $2x^2 - 12x + 25$ and $2x^2 + 12x + 25$, and for what values of x are these achieved?
2. What are the minimum values of $2x^4 - 12x^2 + 25$ and $2x^4 - 12x^2 + 25$, and for what values of x are these achieved?
3. What is the maximum value of $2 + 24 \times 2^x - 3 \times 2^{2x}$? For what value of x is this maximum value achieved?
4. Show that the equation $x^6 + x^2 - 6x + 10 = 0$ has no solutions
5. Find, with proof, all the integers for which $n^2 + 20n + 11$ is a perfect square.
6. Solve the equation $x^4 - 16x^3 + 70x^2 - 48x - 135 = 0$ by writing the left hand side in the form $(\text{quadratic})^2 + \text{c}$.
7. Show that the product of four consecutive integers can never be a perfect square.

Can we complete the cube???

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1. Expand $(x+2)^3$
2. Solve the equation $x^3+6x^2+12x+9=0$
3. Solve the equation $x^3+6x^2+12x=0$
4. Solve the equation $x^3+6x^2+12x+20=0$
5. Expand $(x+a)^3$
6. Write down a cubic equation that can be solved by completing the cube
7. Write down a cubic equation that *can't* be solved by completing the cube

What’s going on geometrically?

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A slice of mathematical history…

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Cardano, Sciopione del Ferro and Tartaglia

If $x^3 + px + q = 0$

(and $4p^3 + 27q^2 > 0$)

Then a solution is $x =$ $\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$

Extra slide

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