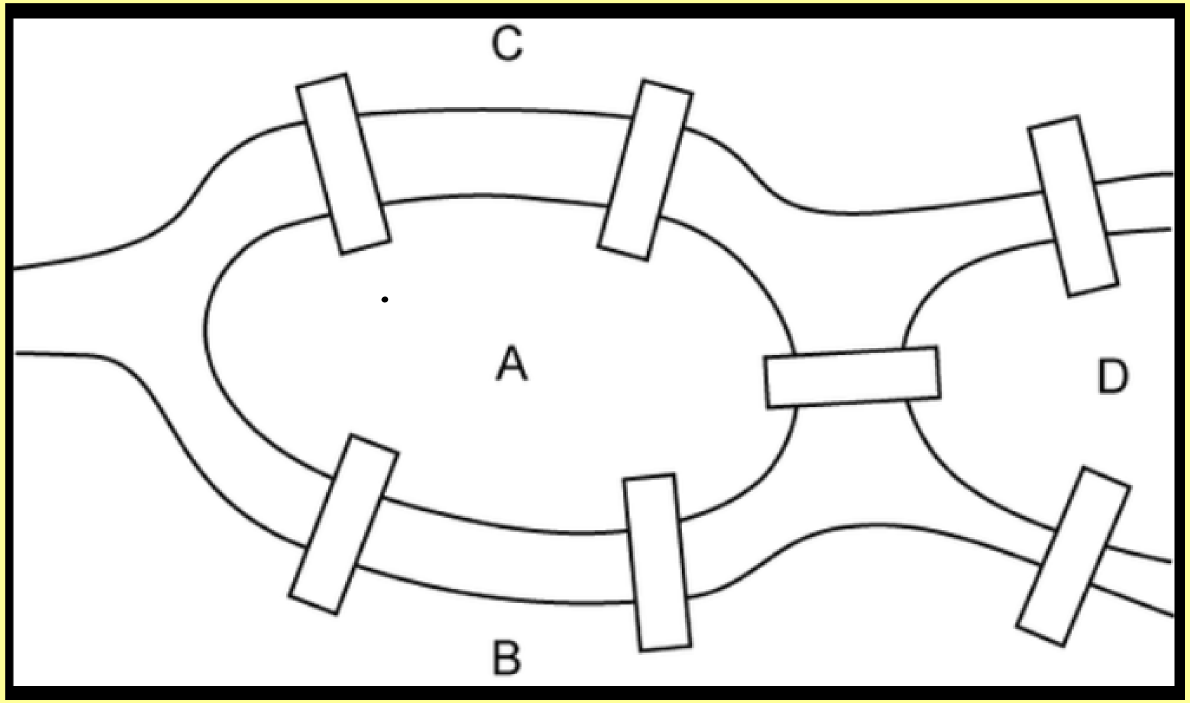
Bridges of Königsberg

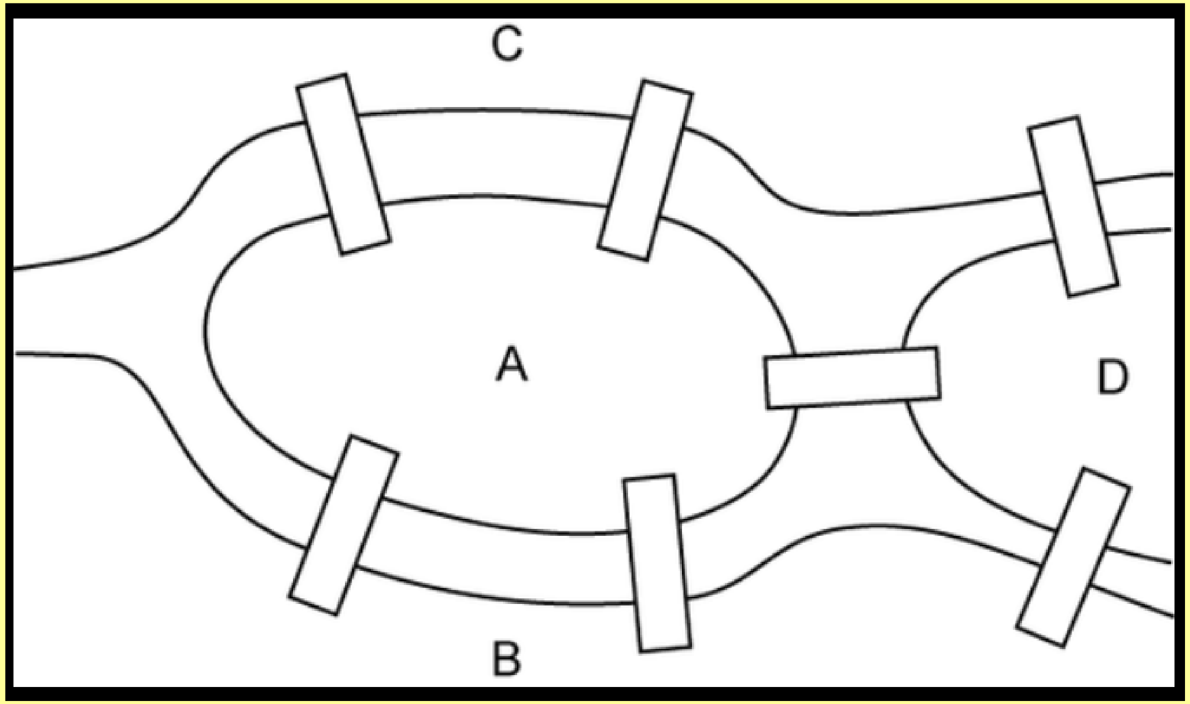
KING'S MATHS SCHOOL

Imagine A, B, C and D are different parts of the city of Königsberg separated by water. There are bridges connecting the different parts of the cities. Can you find a path between the different parts of the cities where you cross all bridges but only once?



Bridges of Königsberg

How can graph theory help us?



Graph Theory Recap

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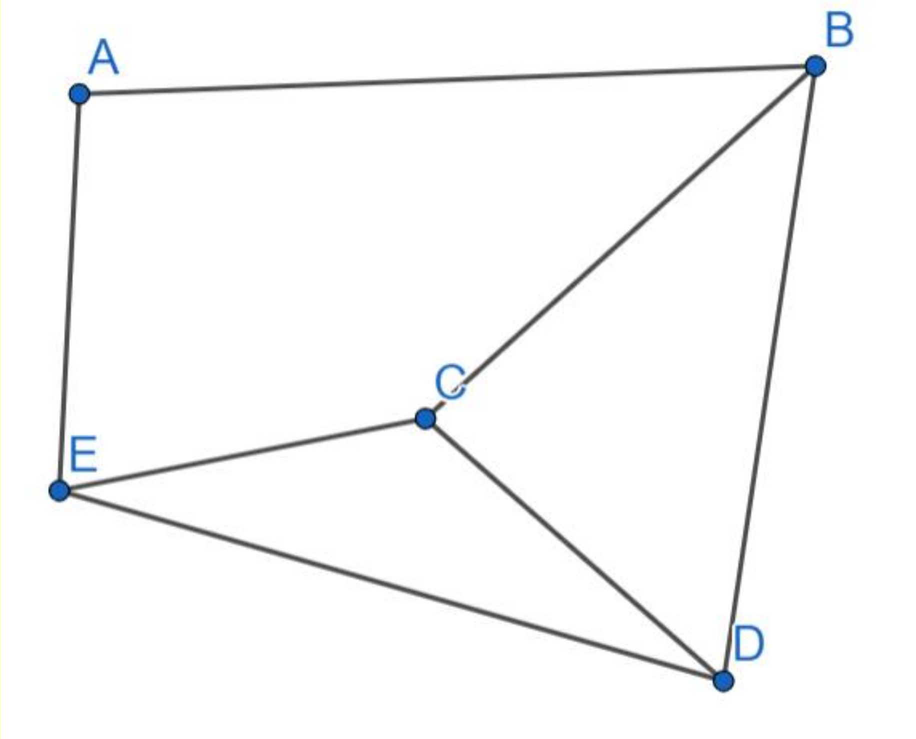


A **graph** G is a structure consisting of a set of vertices V and a set of edges E between different pairs of these vertices.

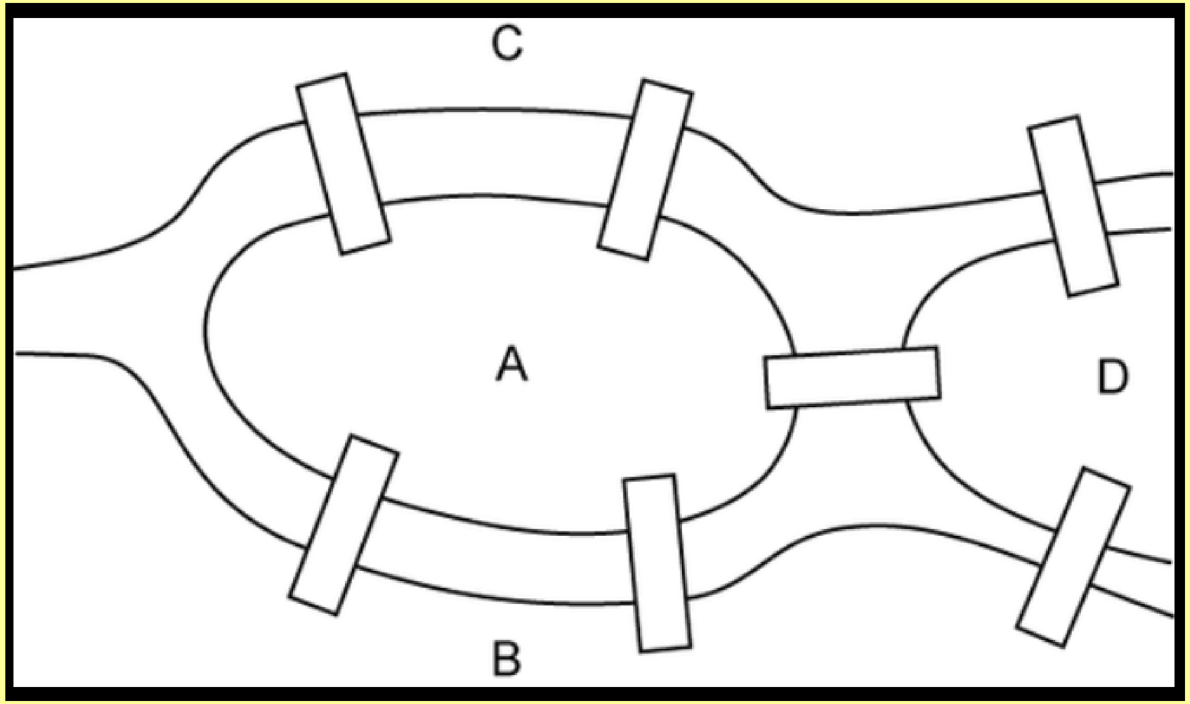
Degree of vertex: Number of edges coming from the vertex.

Degree of graph: Sum of degrees of all vertices in the graph.

Undirected graph: A graph where the order of the two vertices is unimportant; each edge has no direction.

Bridges of Königsberg

How can graph theory help us?



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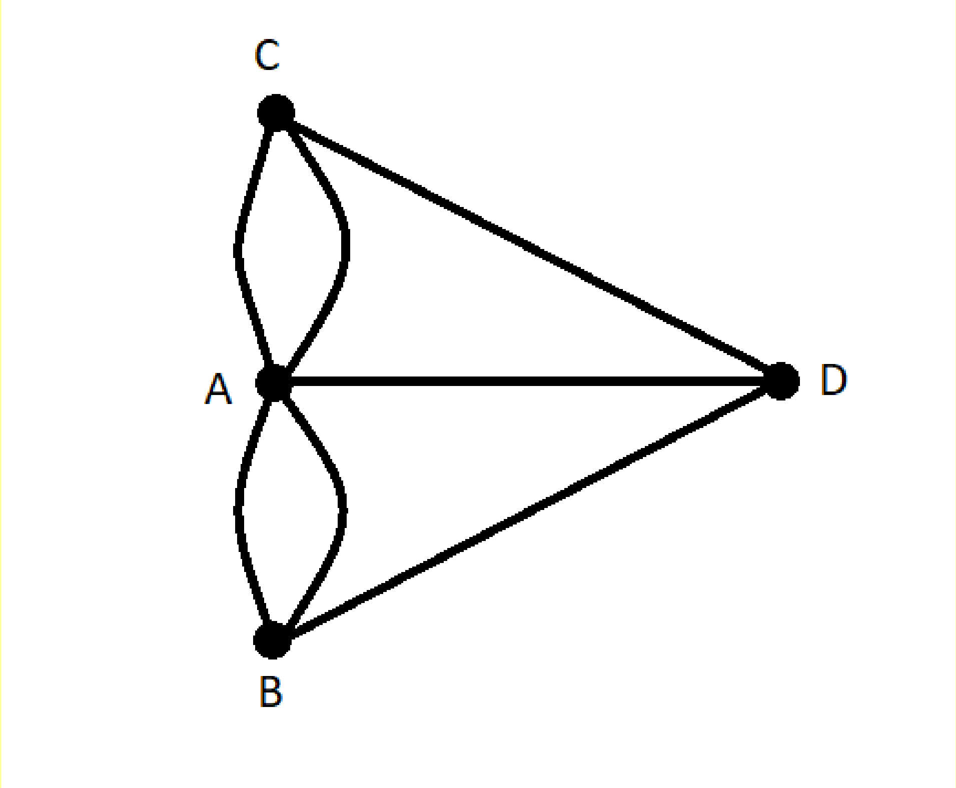
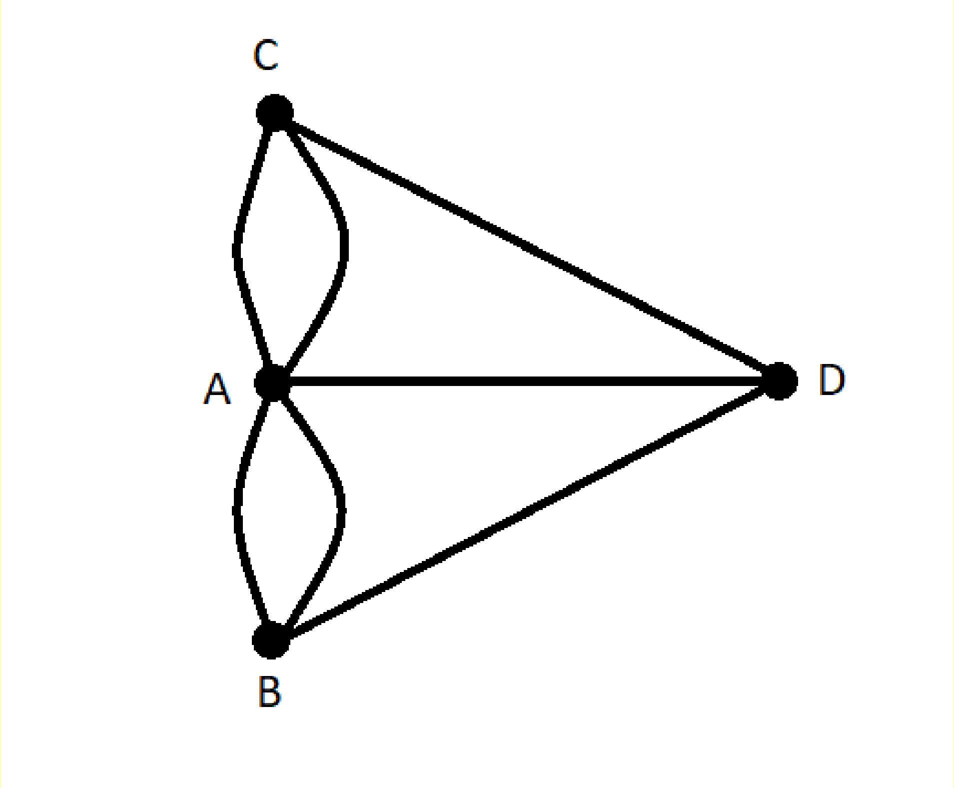


Bridges of Königsberg

How can graph theory help us?

A **graph** G is a structure consisting of a set of vertices V and a set of edges E between different pairs of these vertices.

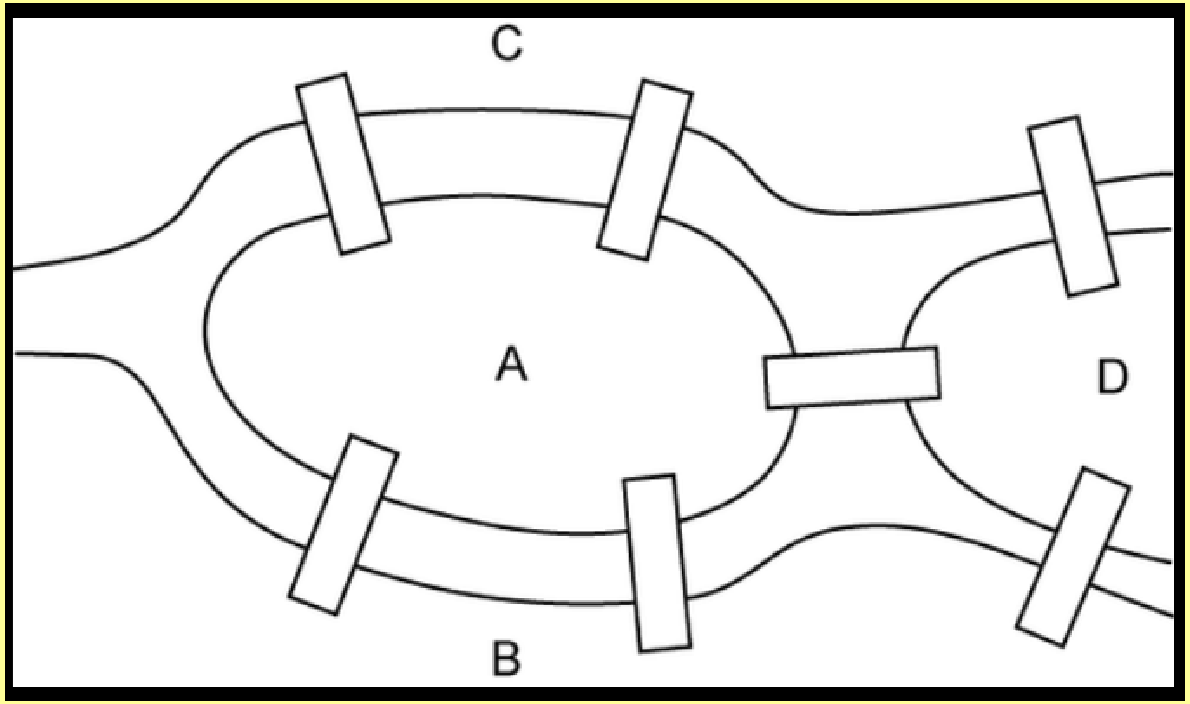


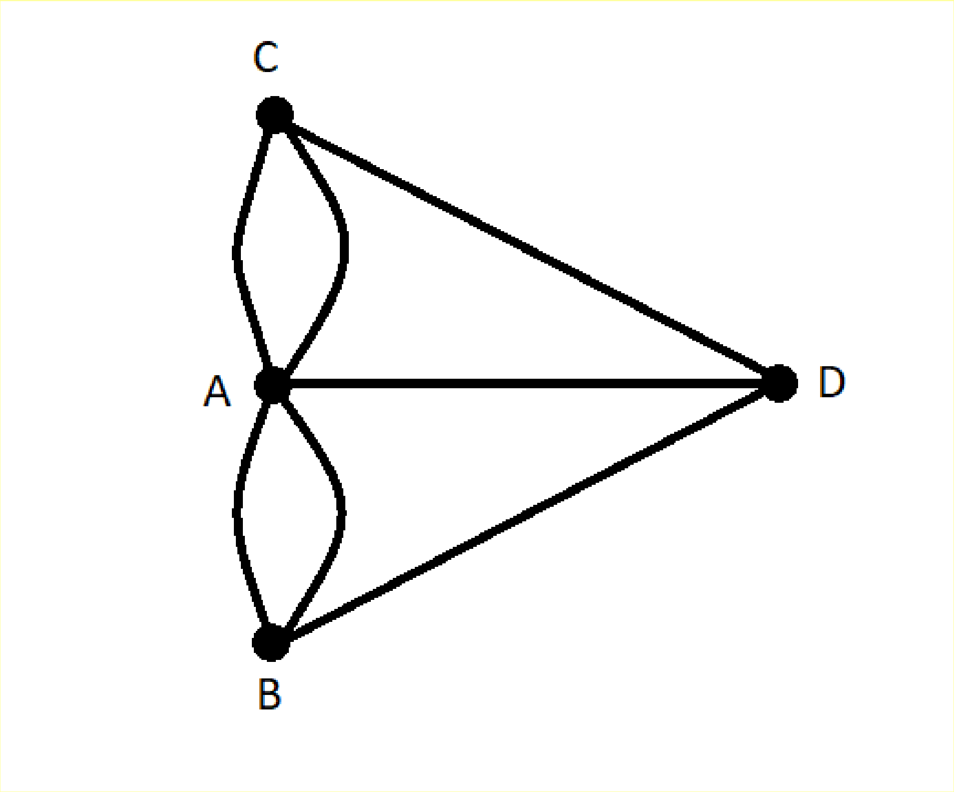
Matrix Notation:

	A	B	C	D
A	0	2	2	1
B	2	0	0	1
C	2	0	0	1
D	1	1	1	0

Euler Paths and Cycles

Euler Path: A path which goes along each edge in the graph exactly once.
Euler Cycle: An Euler path that starts and ends at the same vertex.





Euler Paths and Cycles

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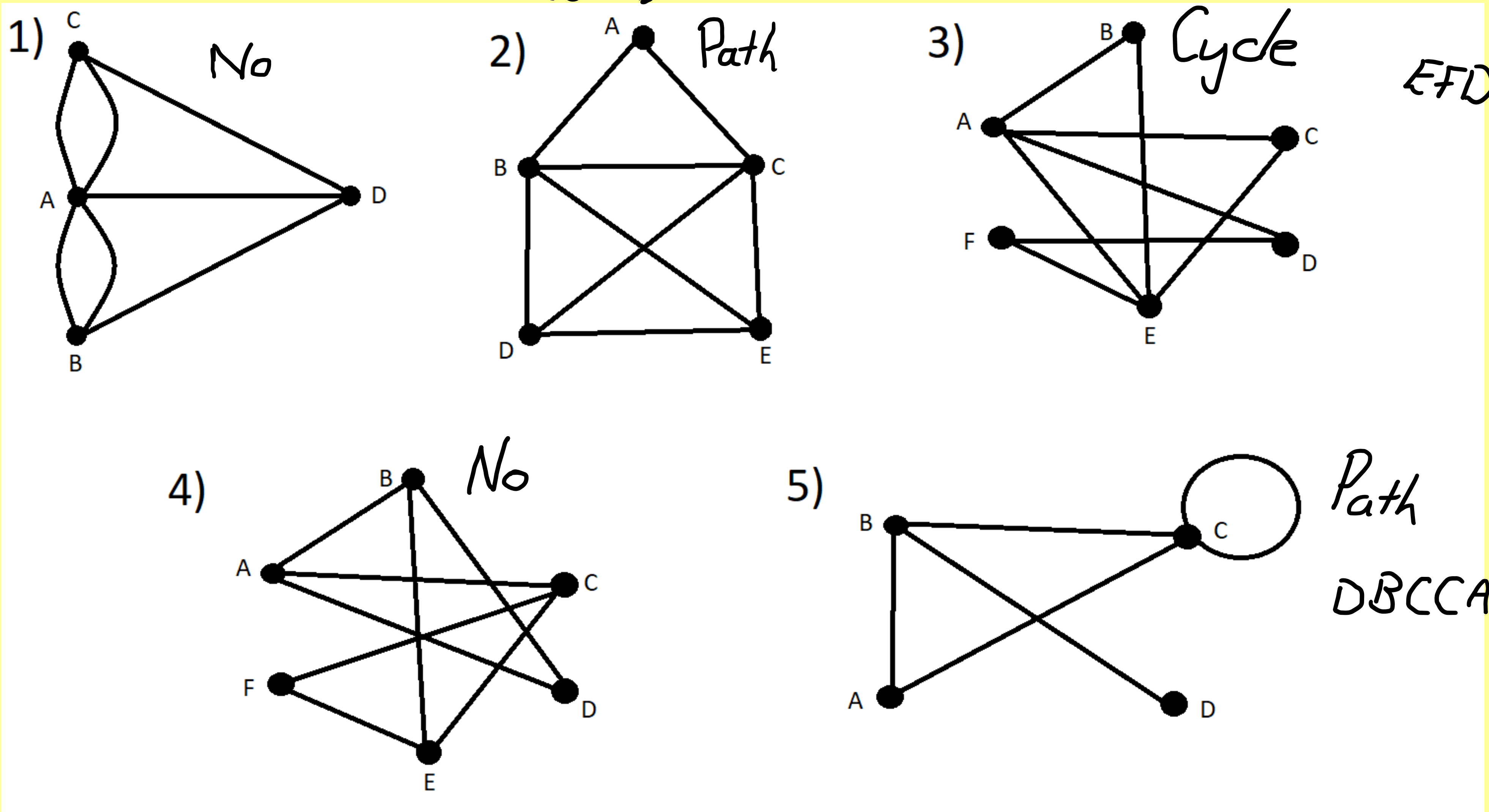
EBCEDBACD

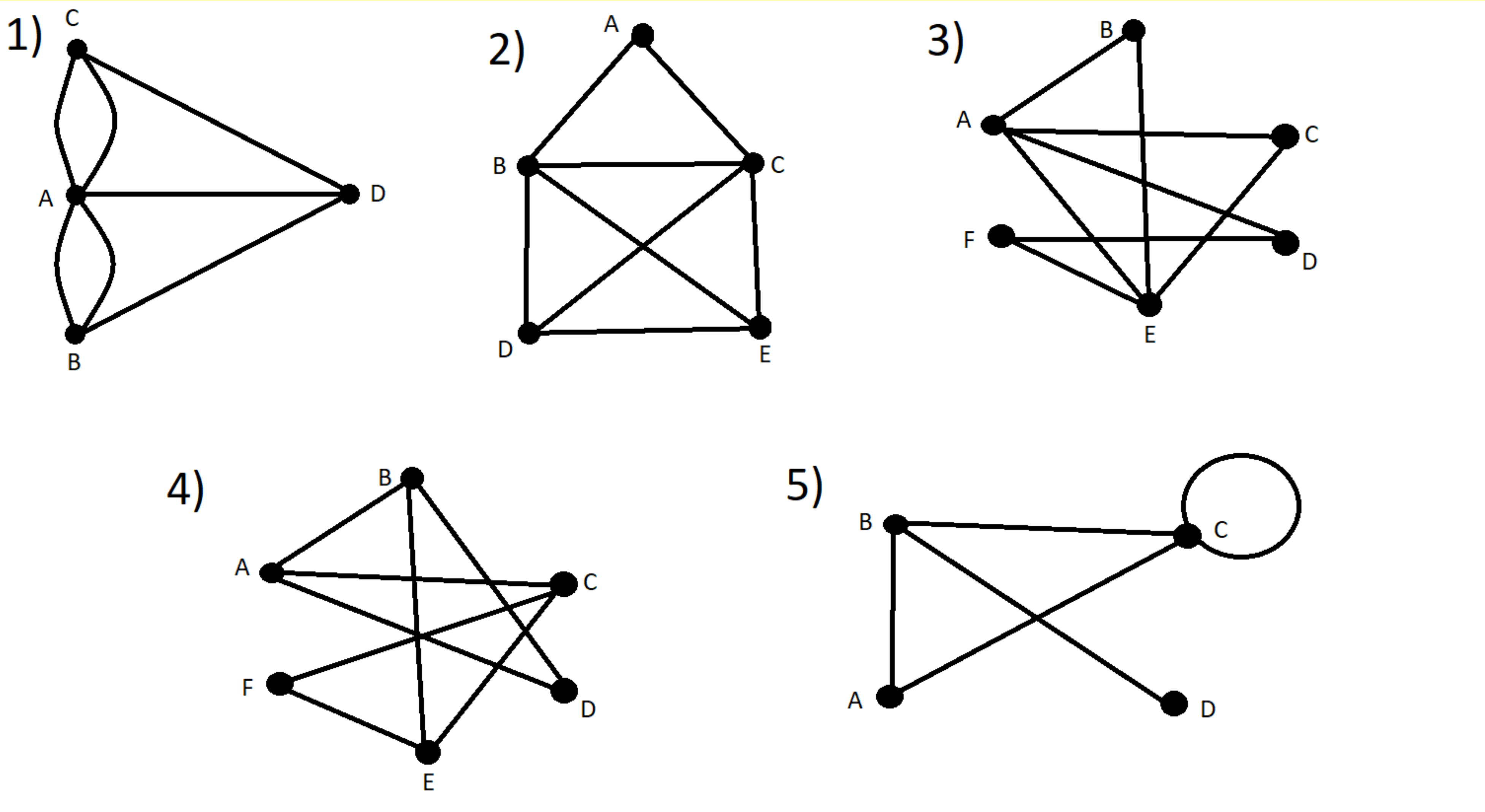


EFDACEABE

DBCCAB

1) Does an Euler path or Euler cycle exist for these graphs?
2) Are there patterns for the graphs which work?

Euler Paths and Cycles



What did you notice?

Look at the degrees of the vertices...

Euler Paths and Cycles

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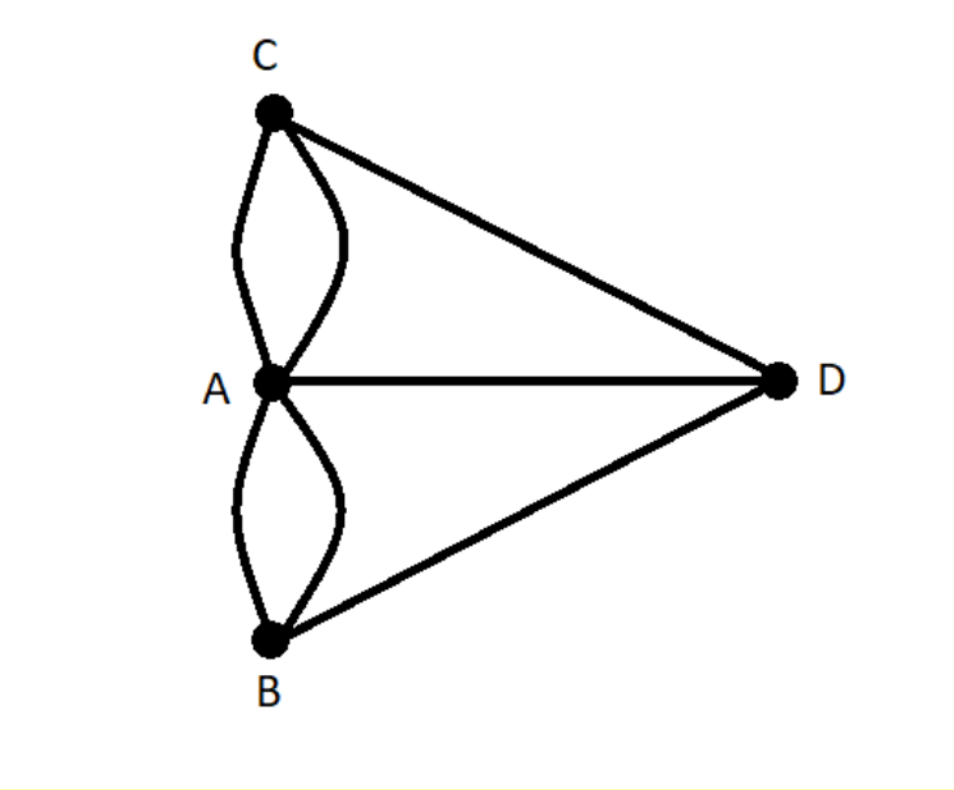
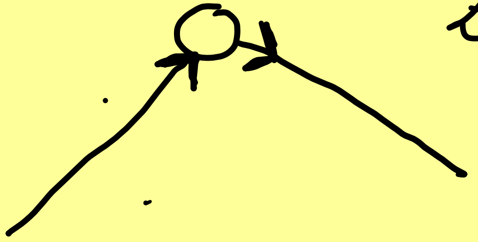
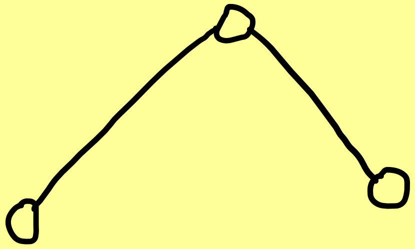
Things you may have noticed:

- If the undirected graph has an Euler cycle, every vertex has even degree
- If the undirected graph has an Euler path exactly zero or two vertices have odd degree





Even
- Edge travelling in
- Edge travelling out



2 vertices of odd degree
- Start at one vertex
- End at another.

Graph Theory

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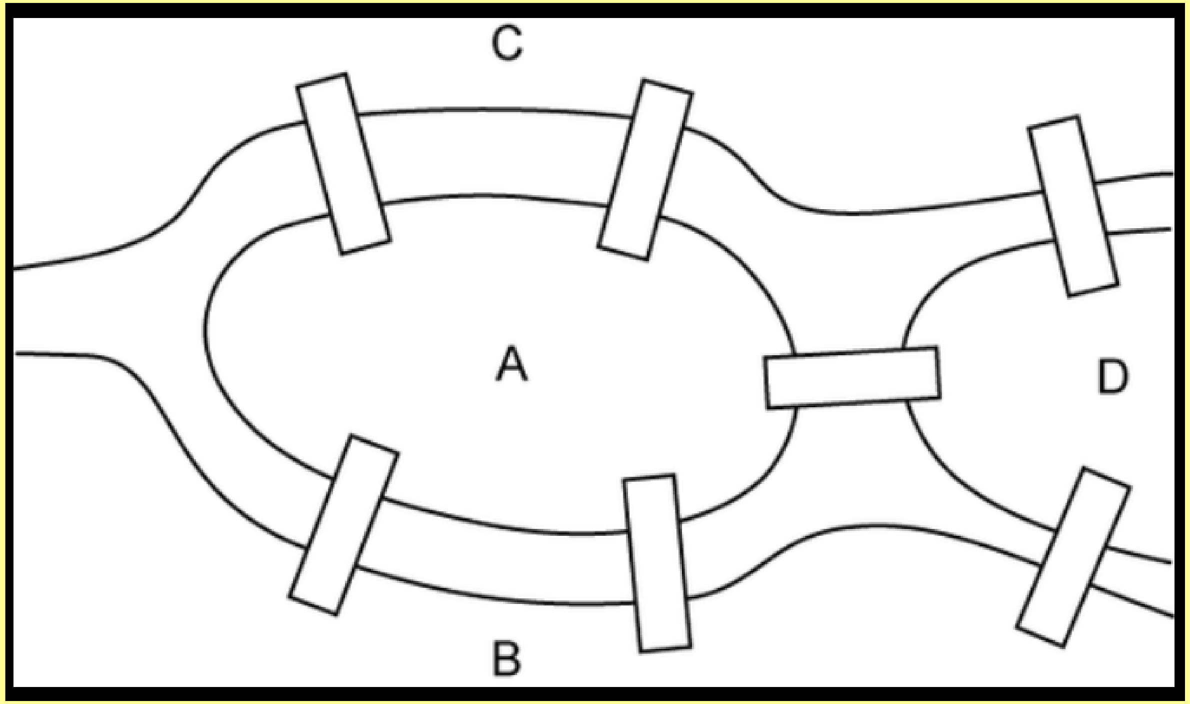
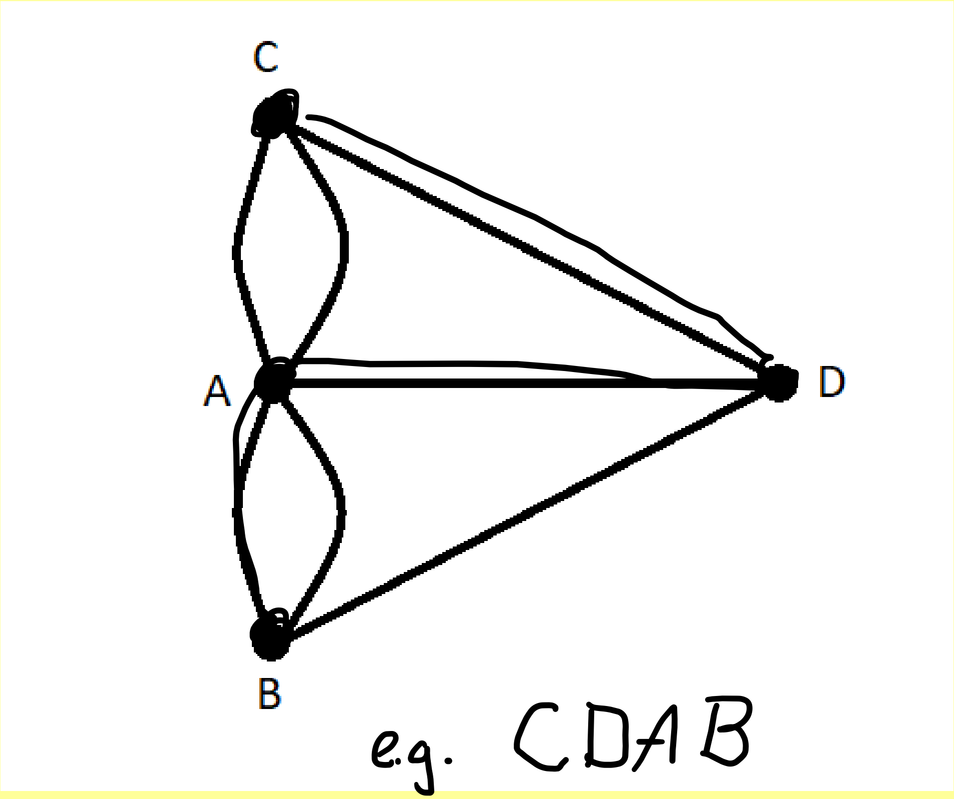
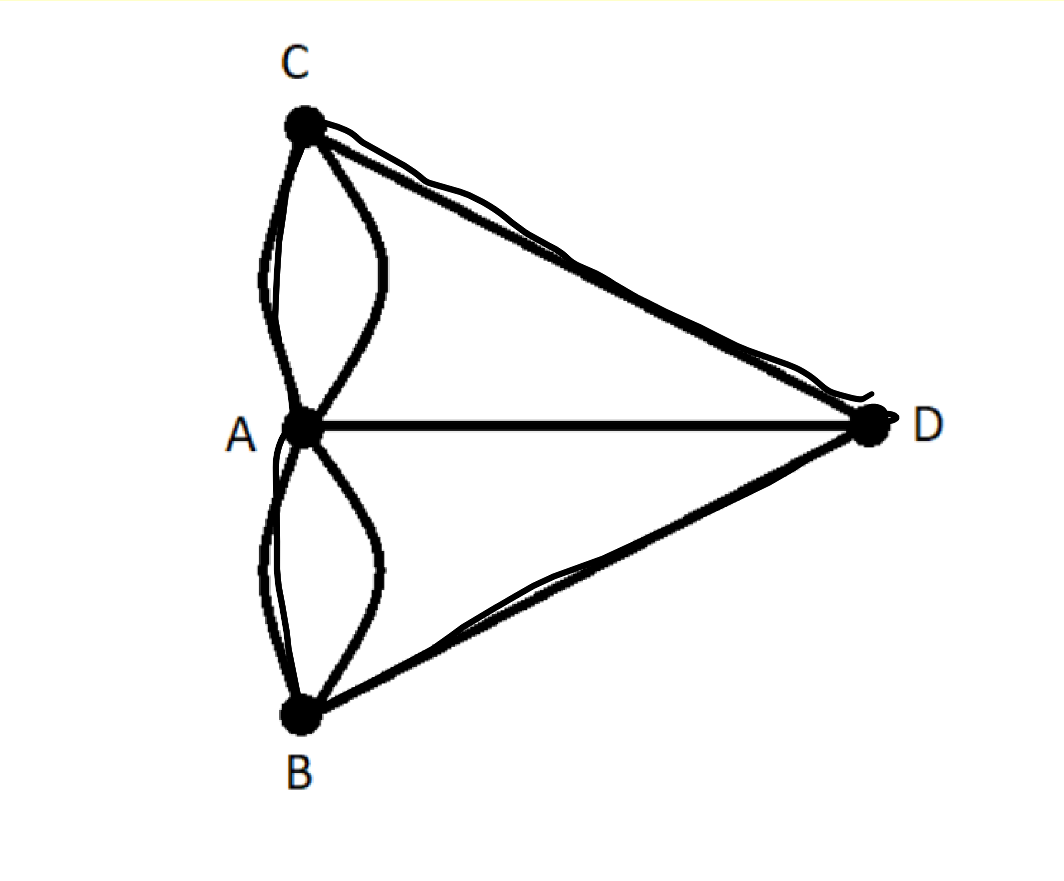
What about if you only want to visit each part of the city once?





Hamiltonian Cycles

Hamiltonian Cycle: A closed loop on the graph where each vertex is visited exactly once.



Eg.

CDBAC

Travelling Salesman Problem

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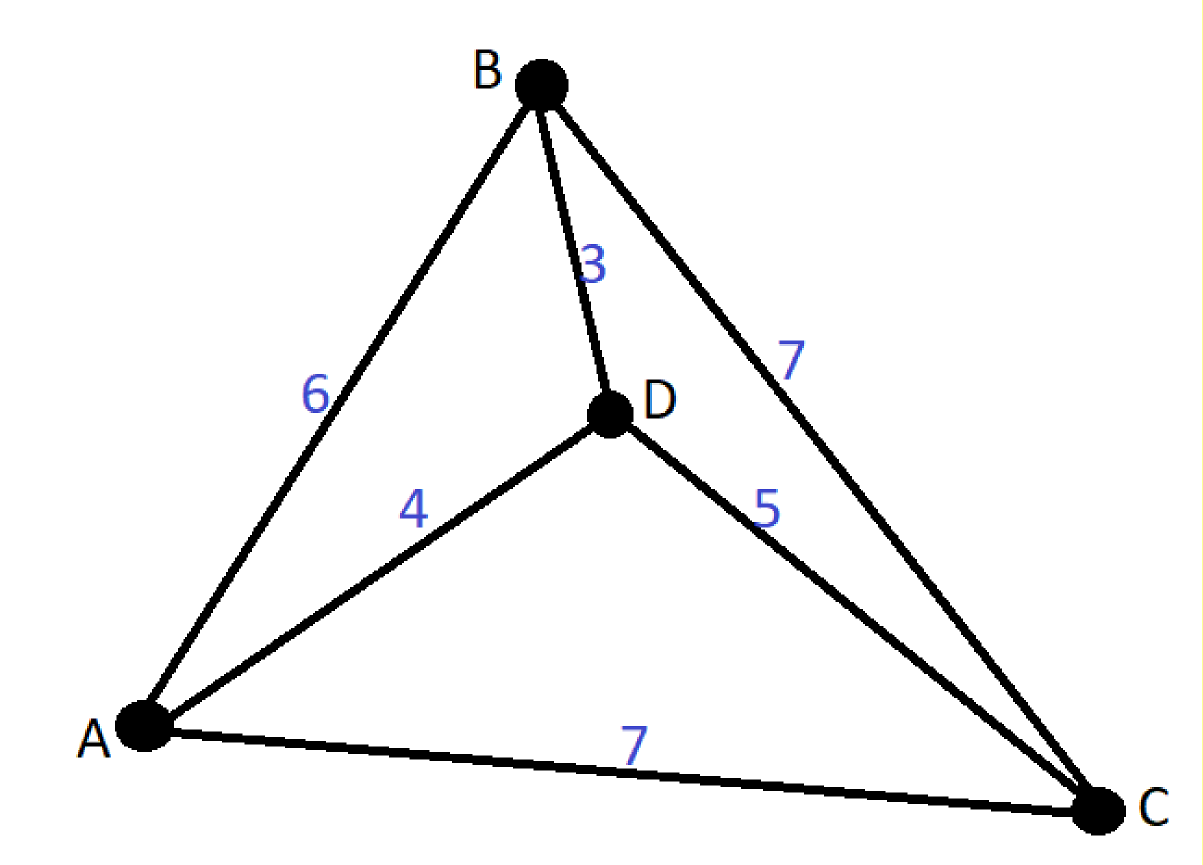
Travelling Salesman Problem: Given a number of cities, and the distance between these cities what is the shortest possible route that visits each city exactly once returning to the original city?

Weighted Graph: A weighted graph is a graph where each edge is given a numerical weight.

In this case, the numerical weight is the distance between the two vertices.



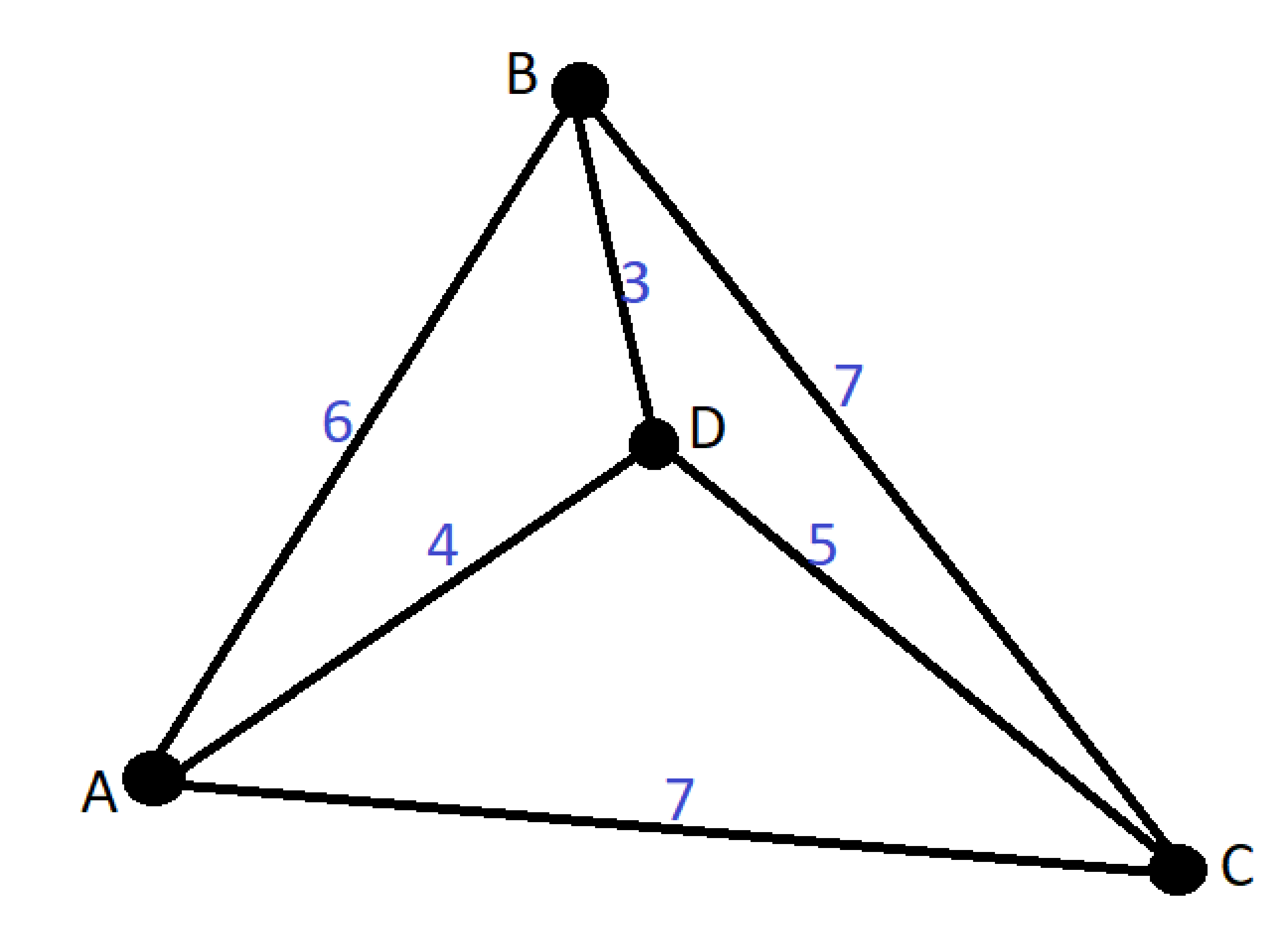
Travelling Salesman Problem

Travelling Salesman Problem: Given a number of cities, and the distance between these cities what is the shortest possible route that visits each city exactly once returning to the original city?

Instructions:

- Starting at A, find the shortest possible distance that is required to visit each of the vertices exactly once and return back to A.
- Can you find a shorter cycle if you are allowed to choose the starting vertex?

Note: you must return back to that vertex to complete the journey.



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KING'S MATHS SCHOOL

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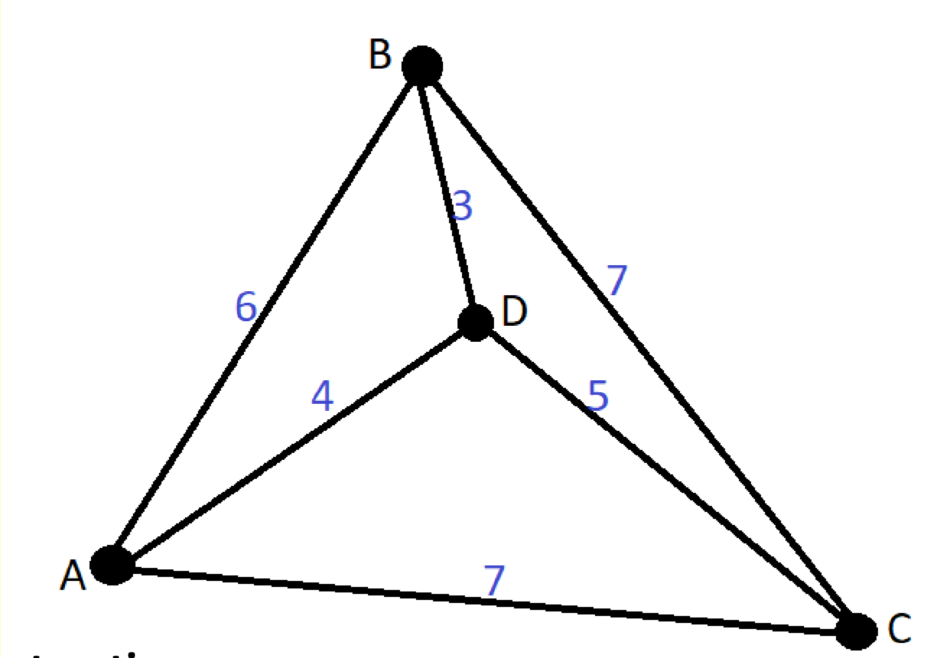

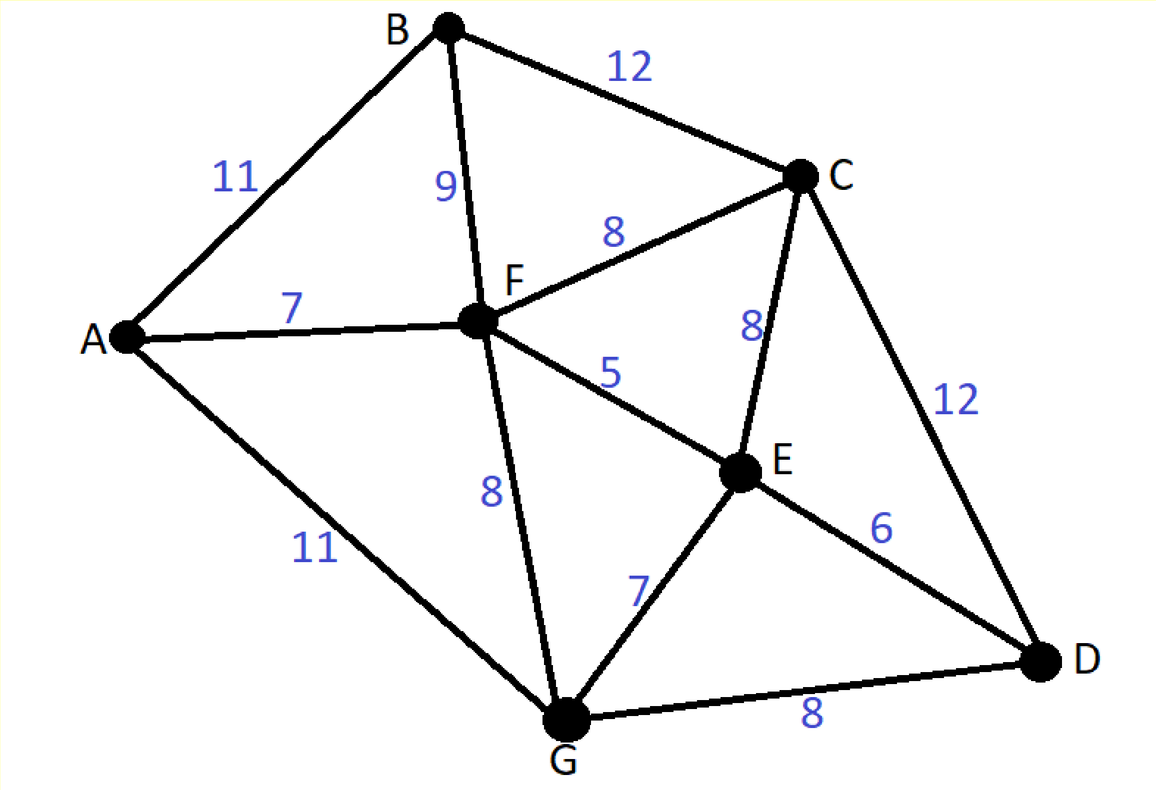

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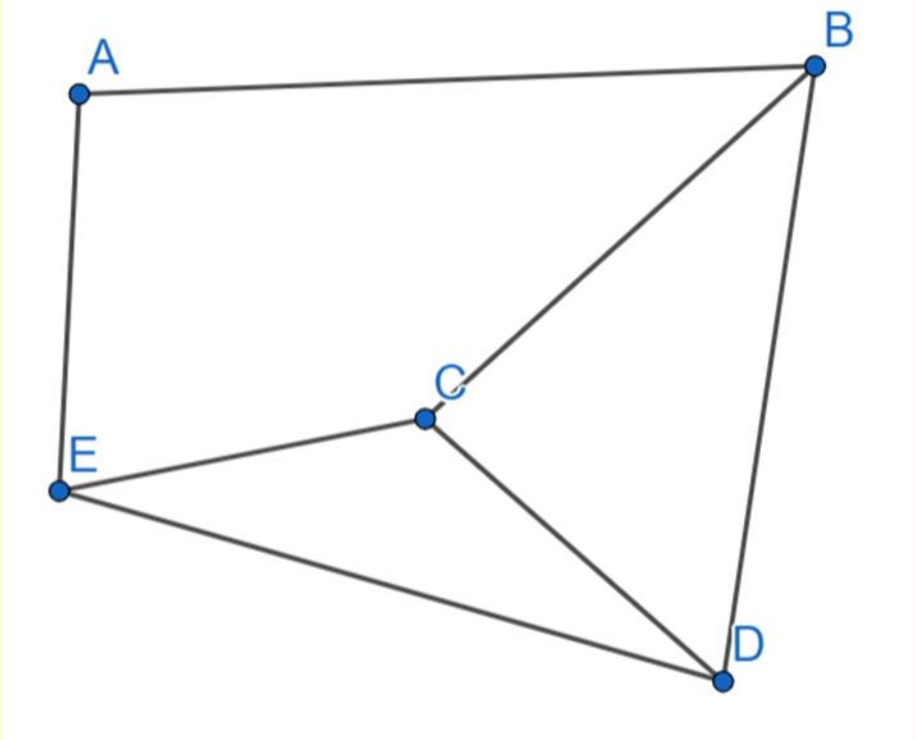
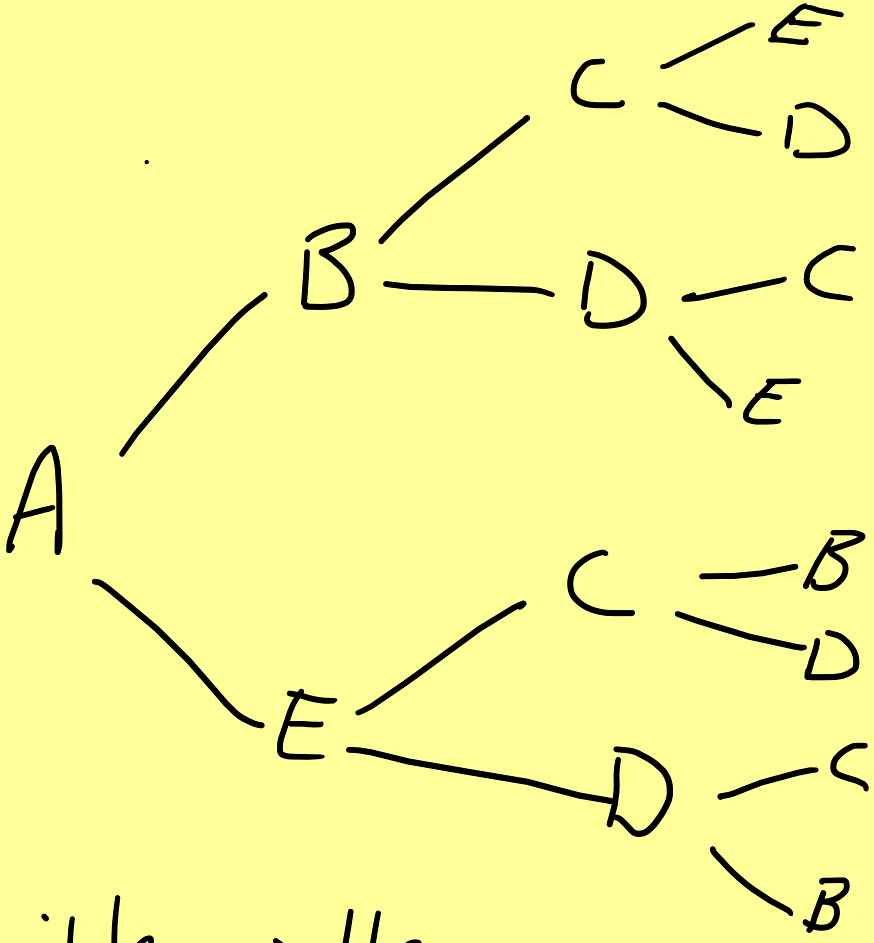
- Starting at A, find the shortest possible distance that is required to visit each of the vertices exactly once and return back to A.
- Can you find a shorter cycle if you are allowed to choose the starting vertex?

Random walks

KING'S MATHS SCHOOL

Starting at A, and randomly walking down any available road but not the one you've just come from, what is the probability of having visited 4 different towns after 3 roads.

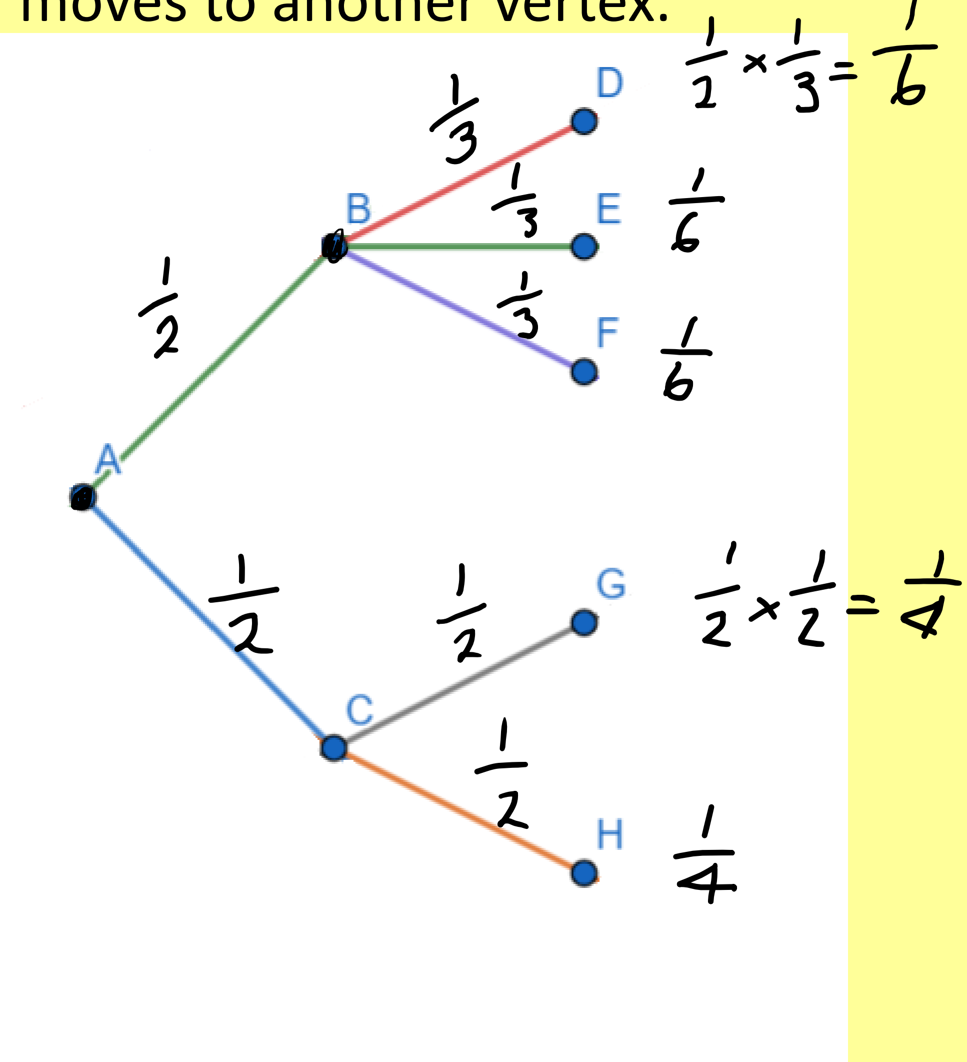




Possible walks

Random walks

A **random walk** on a graph is a process that begins at some vertex, and at each time step moves to another vertex.



ABD 1/6

ABE 1/6

ABF 1/6

ACG 1/4

ACH 1/4

A → F

1/6

Random walks

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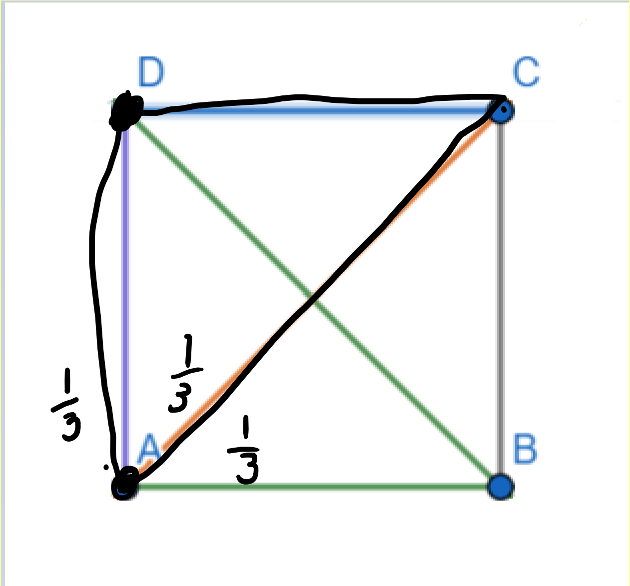
If you follow a random path consisting of travelling down 3 edges, what's the chance of managing to visit four different vertices? If…

a) You randomly choose any edge attached to the vertex at each stage.
b) You cannot walk down the edge you've just walked down.



Random walks

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Start at vertex A. If you follow a random path that travels down 3 edges, what's the chance of managing to visit four **different** vertices? If…

a) You randomly choose any edge attached to the vertex at each stage.
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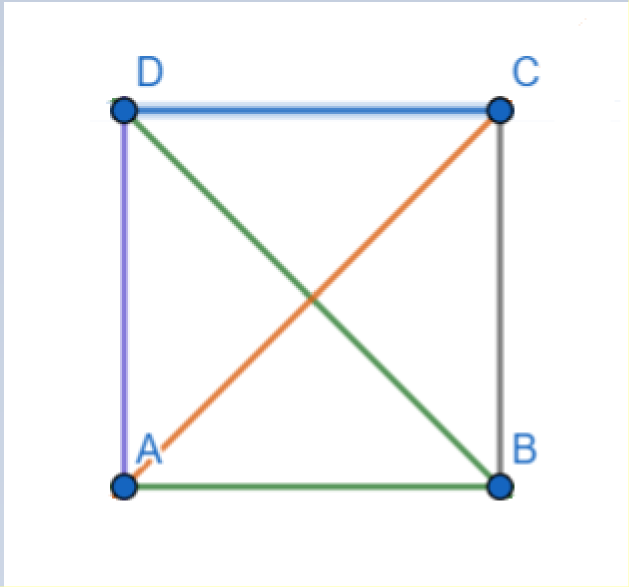
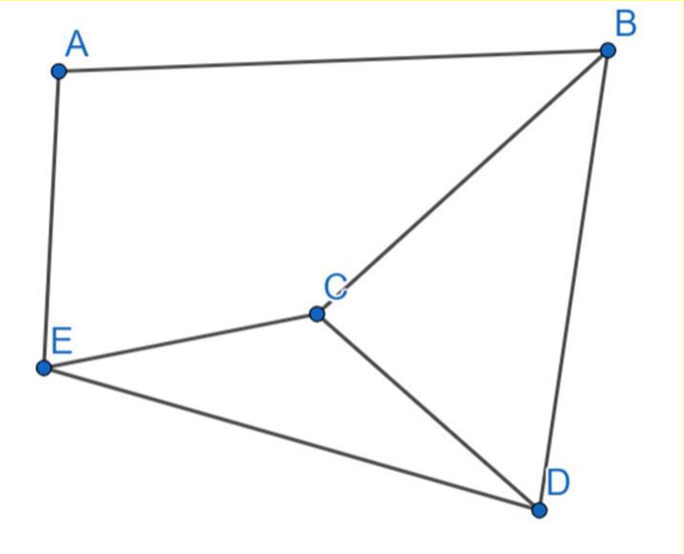
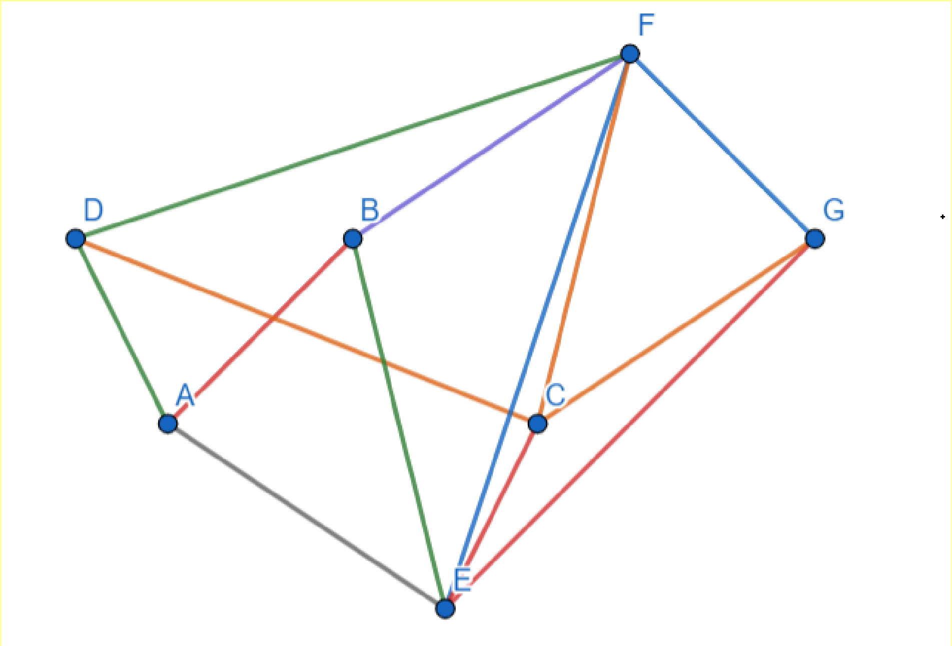
Graph 1



Graph 2



Graph 3



Random walks

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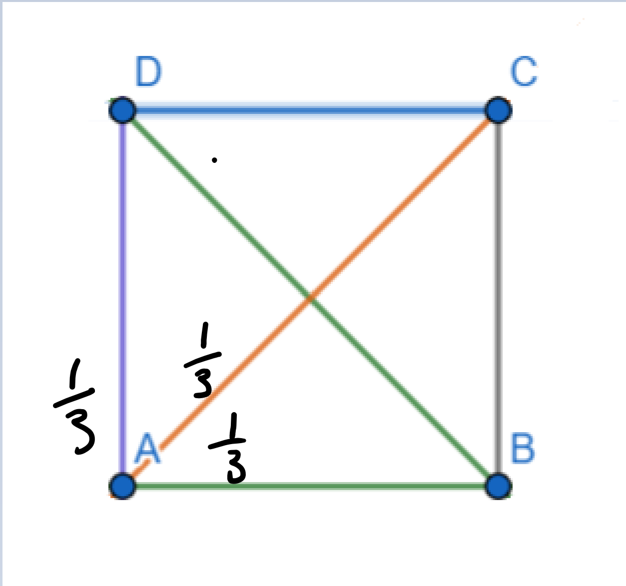

a) $A \xrightarrow{1} B \xrightarrow{\frac{2}{3}} C \xrightarrow{\frac{1}{3}} D$

$1 \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

Probability of visiting a new vertex.

Or

ABCD
ABDC
ACBD
ACDB
ADBC
ADCB

} 6

$6 \times \left(\frac{1}{3}\right)^3 = \frac{6}{27} = \frac{2}{9}$

b)

$6\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)^2 = \frac{6}{12} = \frac{1}{2}$

Or

$A \xrightarrow{1} B \xrightarrow{1} C \xrightarrow{\frac{1}{2}} D$

Random walks with weighted graphs

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Starting at A, and walking down any available road but not the one you've just come from, what is the probability of having visited 4 different towns after 3 roads. You walk down each road with a probability proportional to the weight.

What if we had one way roads?