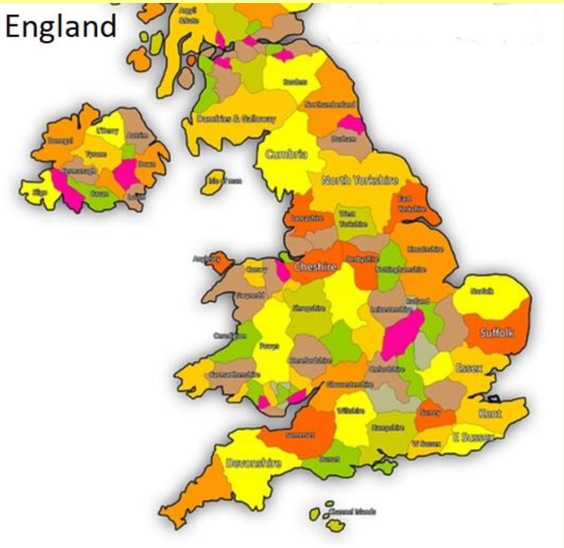
Graph Theory



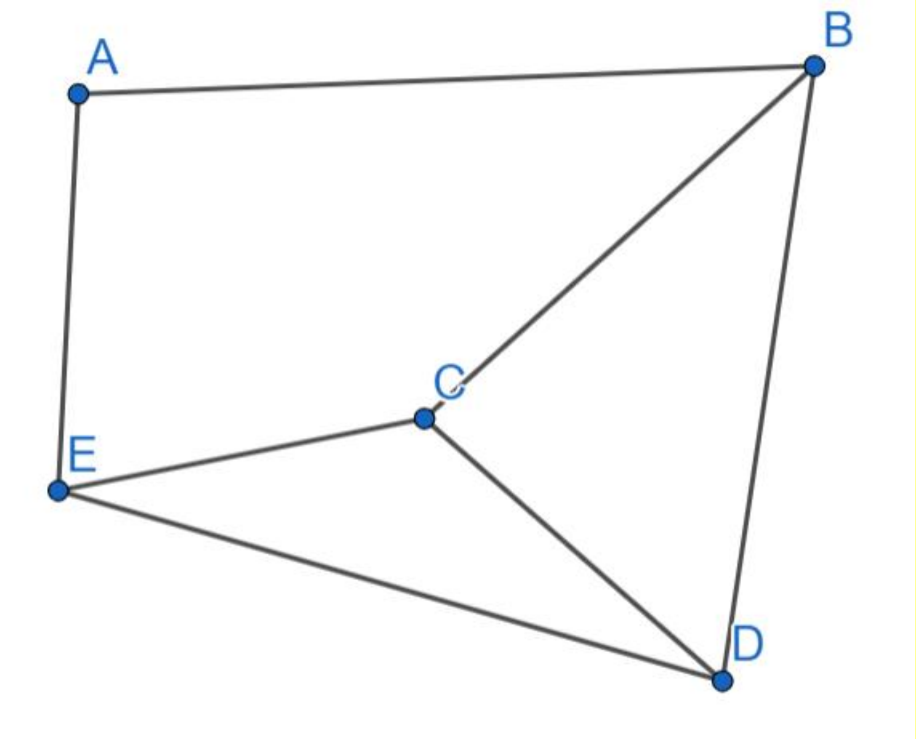
Maps

Social Networks

Neural Connections

What do all these things have in common?

Graph



A **graph** G is a structure consisting of a set of vertices V and a set of edges E between different pairs of these vertices.

What is V here?

What is E?

Graph Example

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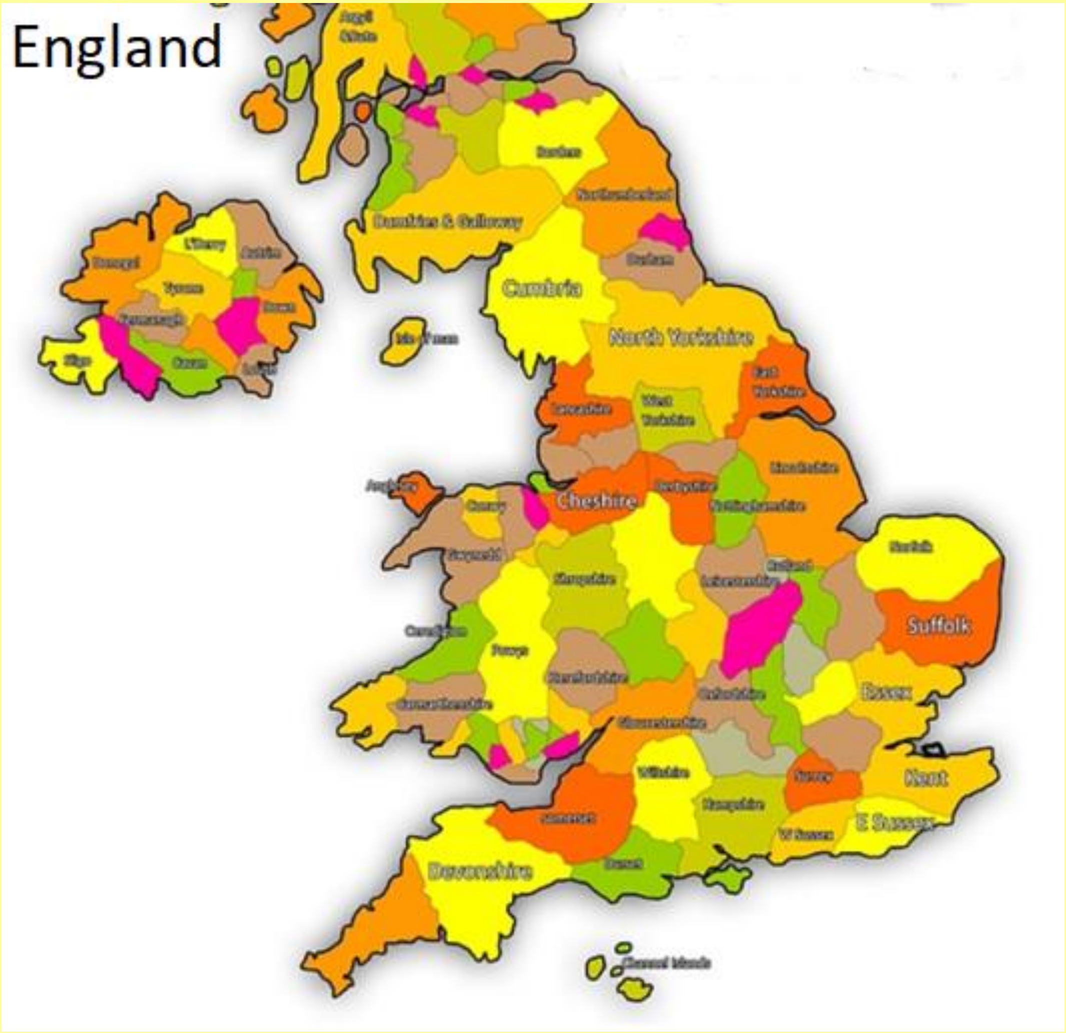

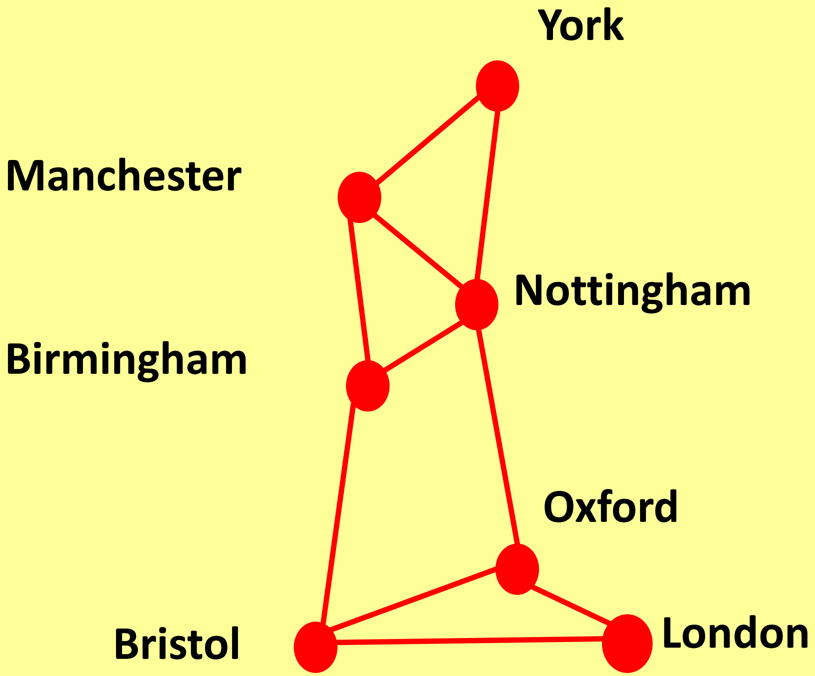

Graph Drawing Notation

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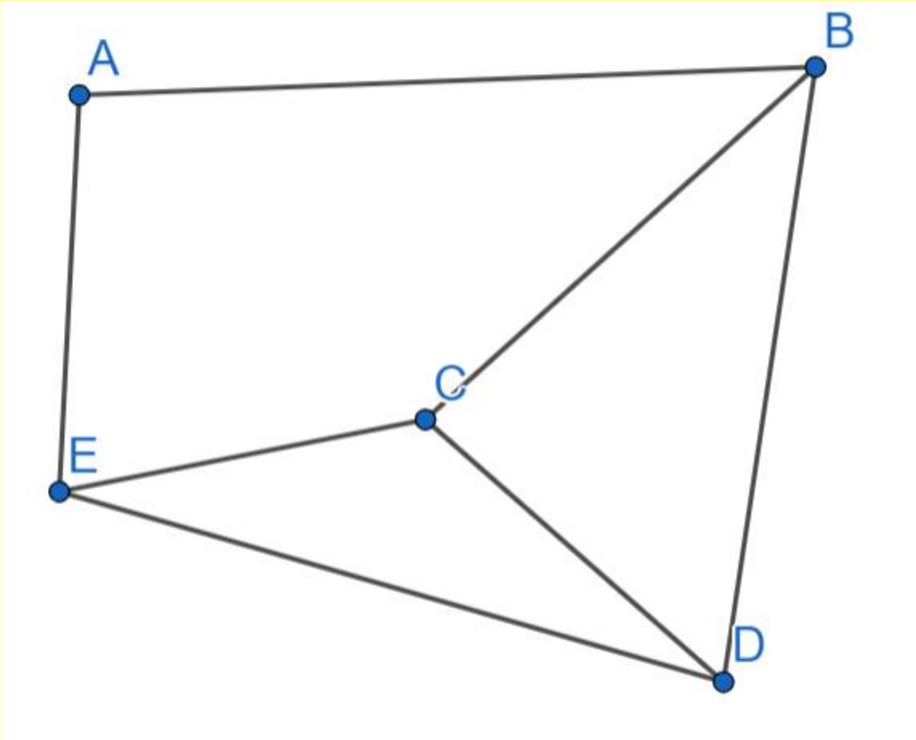


Formal Notation

Let **G**=(V,E).

G is the graph.

V are the vertices

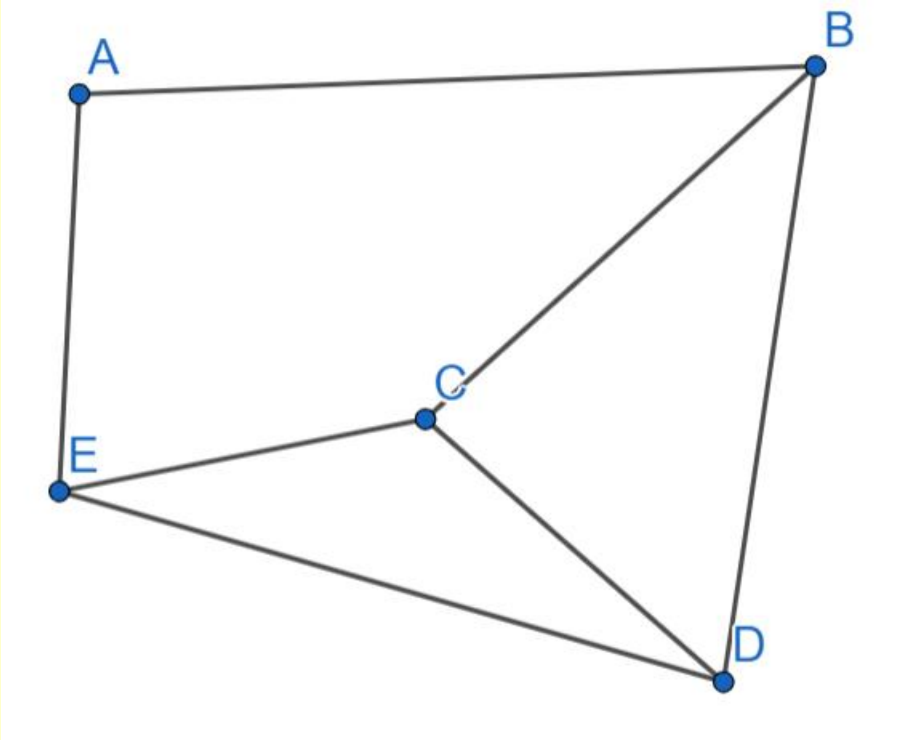
E are the edges.

V={A, B, C, D}

An edge example: {A,B}

E={ {A,B}, {A,E}, {B,C}, {B,D}, {C,D}, {C,E}, {D,E}}

Graph Drawing Notation



Matrix Notation

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	0
C	0	1	0	1	1
D	0	1	1	0	1
E	1	0	1	1	0

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

Graph Drawing

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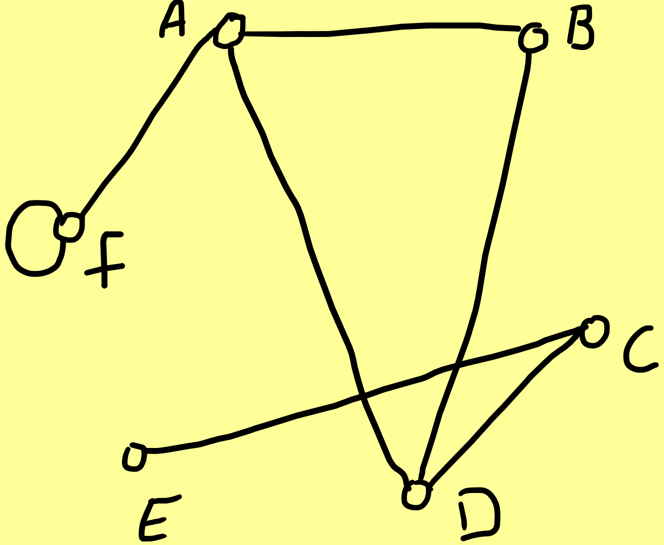
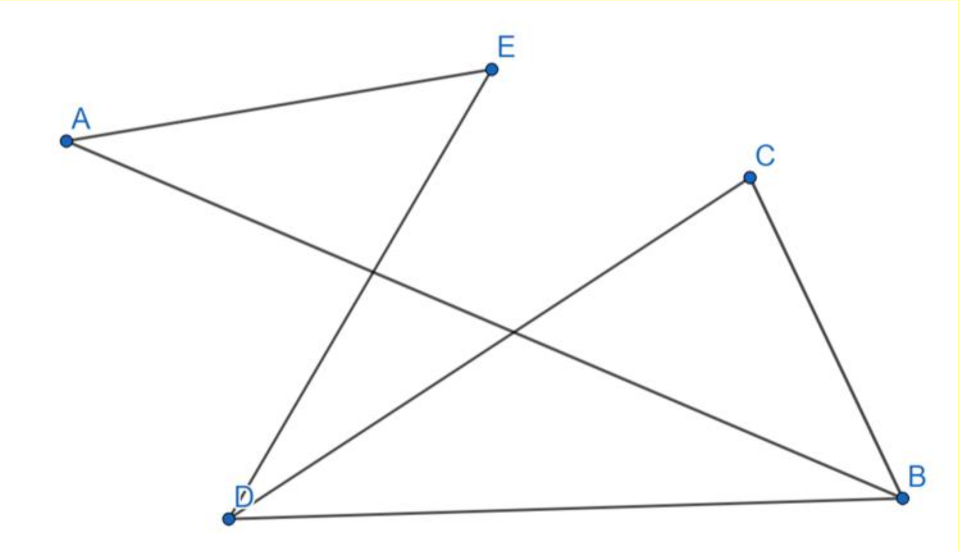
Can you draw the following graph from the matrix:

	A	B	C	D	E	F
A	0	1	0	1	0	1
B	1	0	0	1	0	0
C	0	0	0	1	1	0
D	1	1	1	0	0	0
E	0	0	1	0	0	0
F	1	0	0	0	0	1

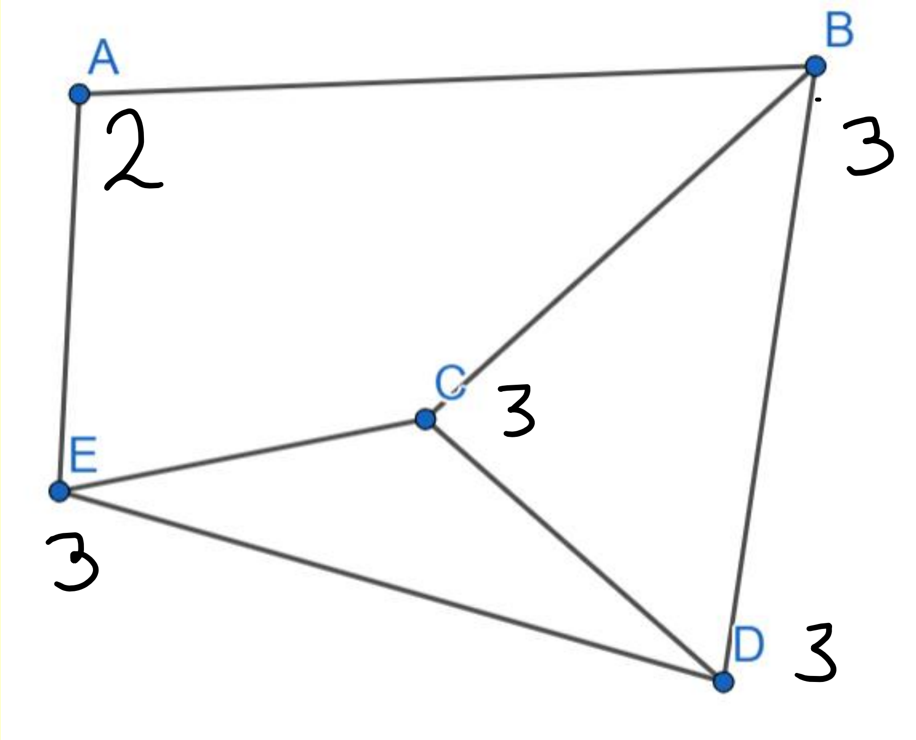


Can you write the matrix notation for the following graph:



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Properties of Graphs: Degree



Degree: The degree of a vertex is the number of connections coming from the vertex.

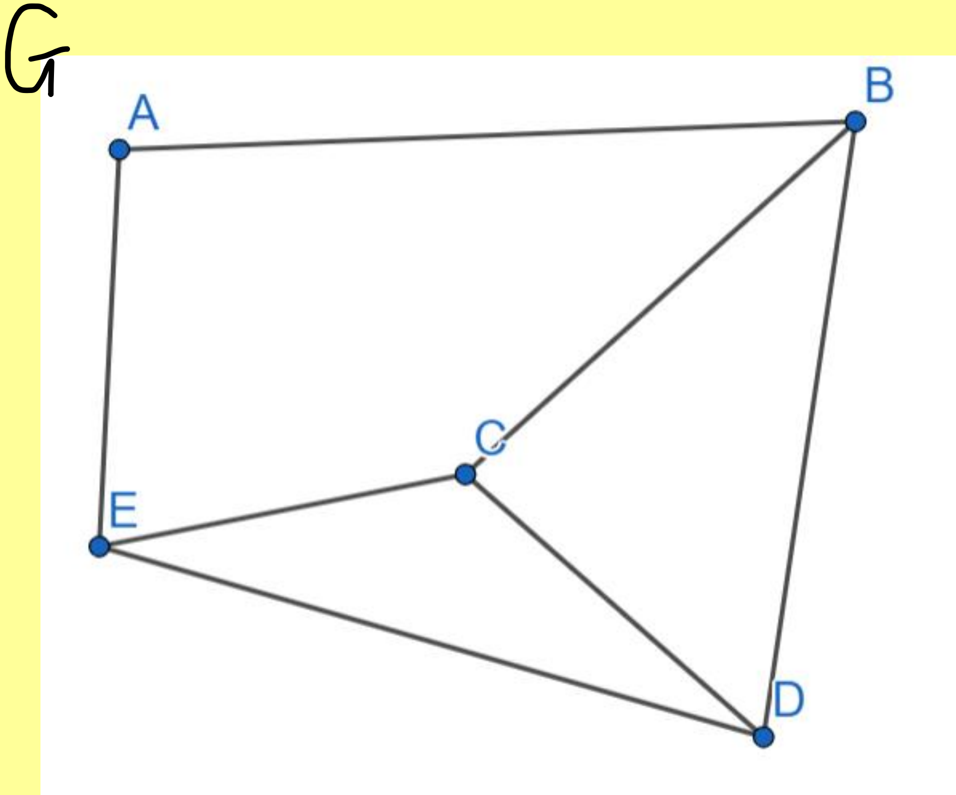
Notation: deg(B) for degree of vertex B.

For graph G, deg(G) is the sum of the degrees of all vertices in the graph.

What it deg(G) for the given graph?

Degree of graph: 2+3+3+3+3=14

Properties of Graphs: Degree



$\Delta(G) = 3$

$\delta(G) = 2$

Degree: The degree of a vertex is the number of connections coming from the vertex.

Notation: deg(B) for degree of vertex B.

For graph G, deg(G) is the sum of the degrees of all vertices in the graph.

$\Delta(G)$= Maximum degree of a graph, G.

$\delta(G)$= Minimum degree of a graph, G.

Properties of Graphs: Degree

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Consider this:

Imagine you have a group of 6 people.

4 of them are all friends with each other.

The 5th person is friends with two of those original 4 people.

The final person is friends with none of the other 5 people.

How might we use a graph to model this?

Properties of Graphs: Degree

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Imagine you have a group of 6 people.

4 of them are all friends with each other.

The 5th person is friends with two of those original 4 people.

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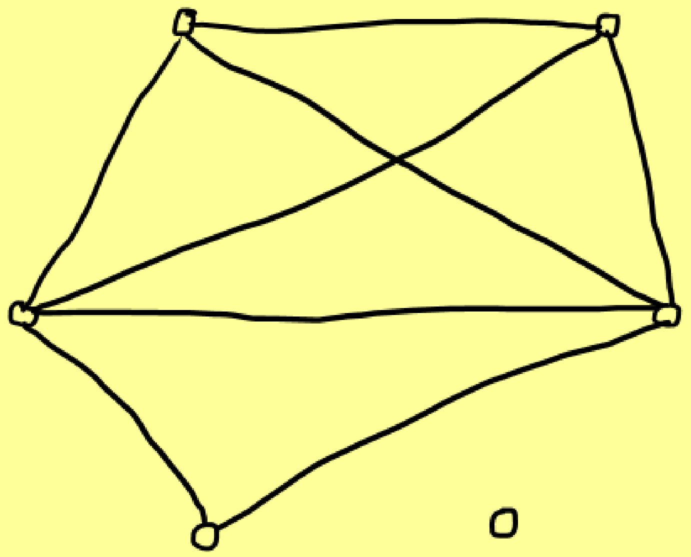
Graph



Graph Theory

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Isolated vertex: A vertex which is connected to no other vertices (i.e. a vertex with degree zero).

Connected: A graph is connected if for each vertex you can get to any other vertex by travelling along a series of edges i.e there is a path between each pair of vertices.

Connected component: A set of (maximum) vertices that are connected.

Clique: A subset of vertices where each pair of vertices is connected by an edge.

Graph Theory Questions

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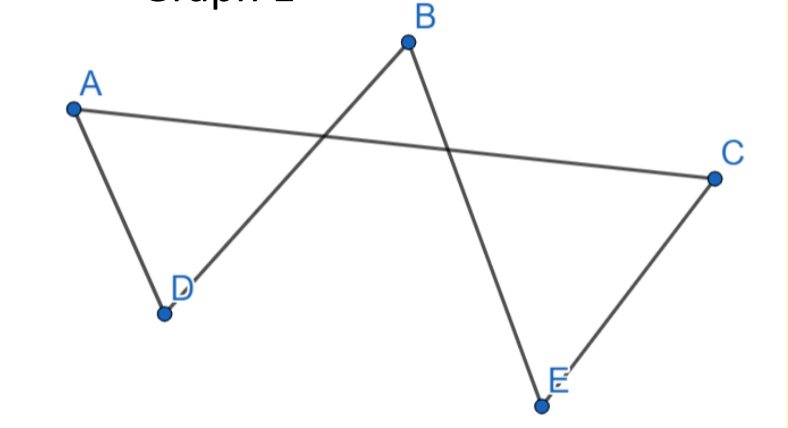
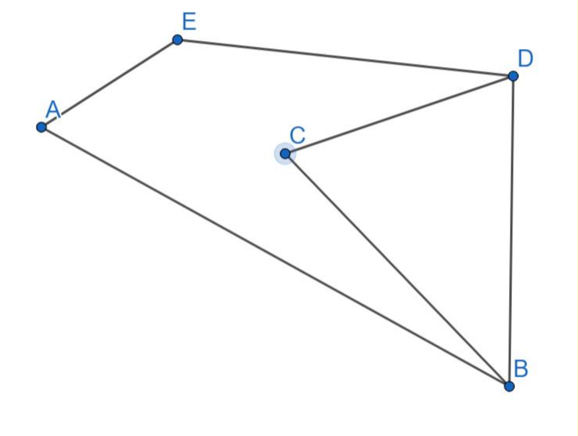
1) **Can you write the following graphs in matrix notation?**
- **How many vertices/ edges does each graph have?**
- **What is the total degree and maximum/ minimum degree of each graph?**
- **What is the average degree of each graph?**

Do you notice any patterns between the number of edges, vertices, degrees etc?

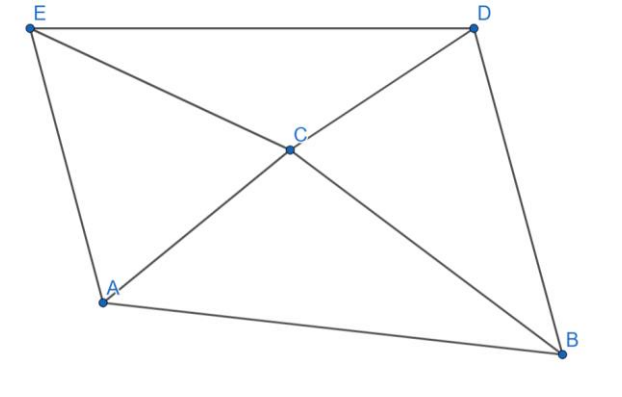
Graph 1



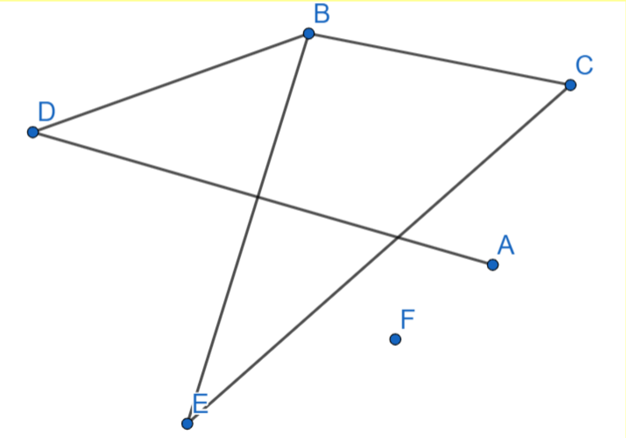
Graph 2



Graph 3



Graph 4



Graph Theory Questions

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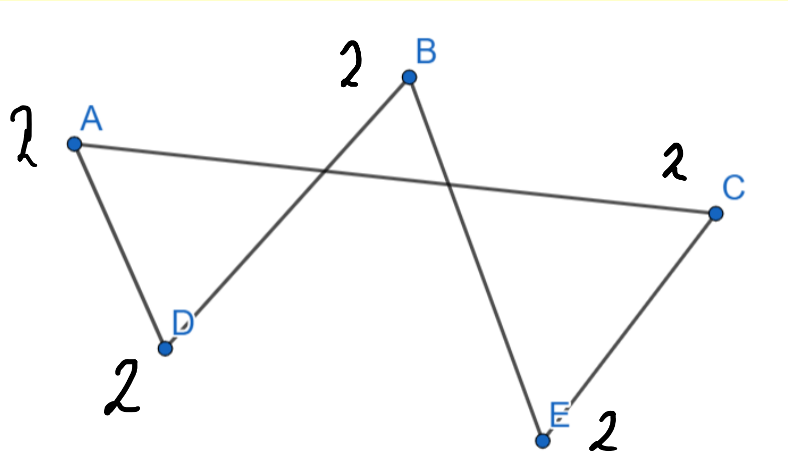
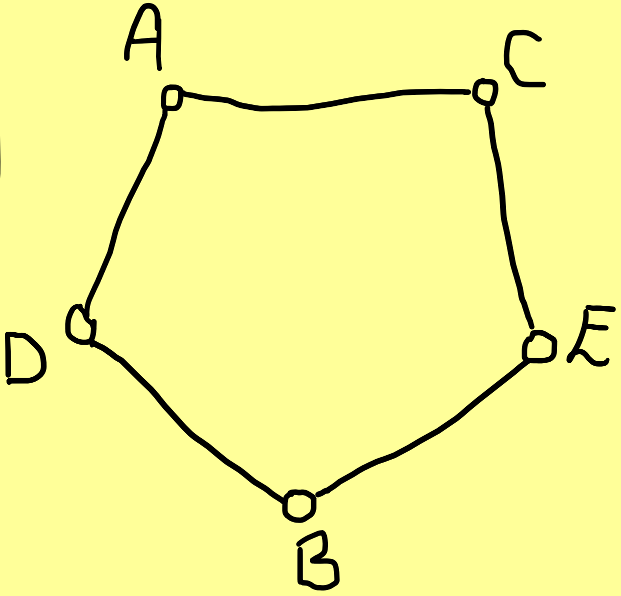
Graph 1

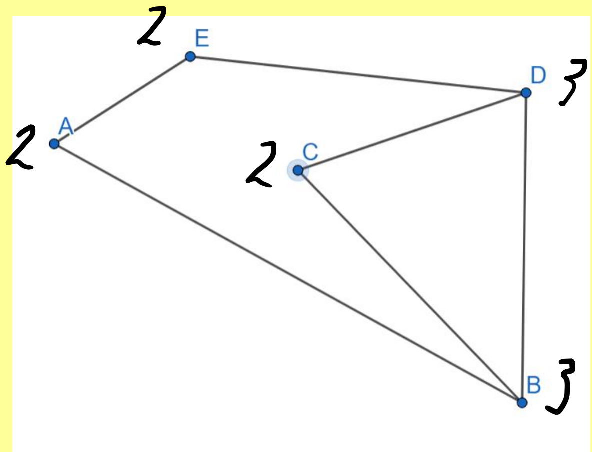


	A	B	C	D	E
A	0	0	1	1	0
B	0	0	0	1	1
C	1	0	0	0	1
D	1	1	0	0	0
E	0	1	1	0	0



Total degree: 10

Graph 2



	A	B	C	D	E
A	0	1	0	0	1
B	1	0	1	1	0
C	0	1	0	1	0
D	0	1	1	0	1
E	1	0	0	1	0

Total degree: 12

Graph Theory Questions

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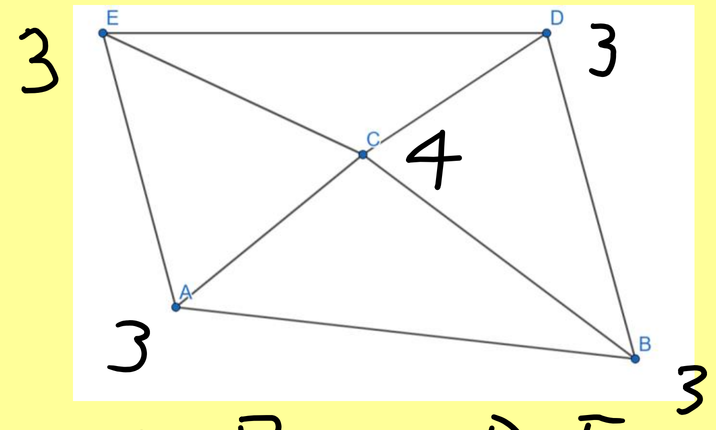
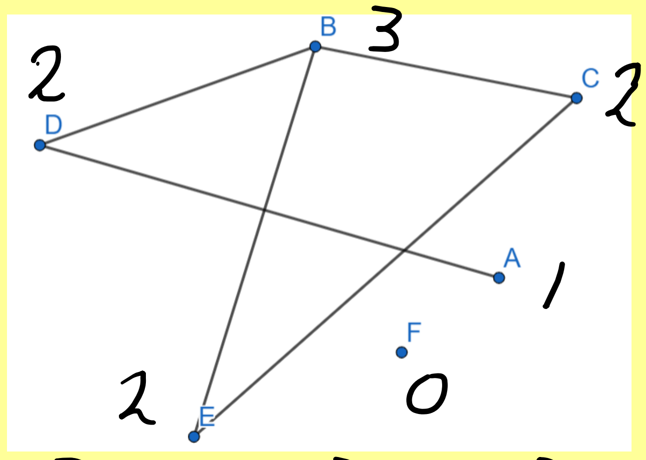
Graph 3



	A	B	C	D	E
A	0	1	1	0	1
B	1	0	1	1	0
C	1	1	0	1	1
D	0	1	1	0	1
E	1	0	1	1	0

Total degree: 16

Graph 4



	A	B	C	D	E	F
A	0	0	0	1	0	0
B	0	0	1	1	1	0
C	0	1	0	0	1	0
D	1	1	0	0	0	0
E	0	1	1	0	0	0
F	0	0	0	0	0	0

Total degree: 10

Graph Theory Questions

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2) Based on the graph notation, draw the following graphs:

Graph 1

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Graph 2

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Graph 3

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Graph 4

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

a) **What is the average and minimum and maximum degree of each graph?**
b) **How many connected components are there?**
c) **What's the size of the largest clique?**

Graph Theory Questions

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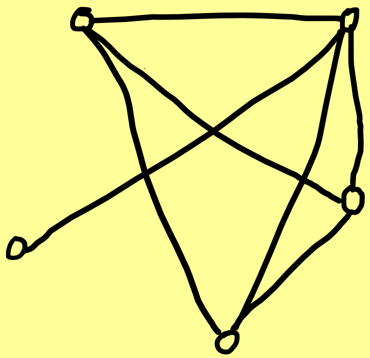
2) Based on the graph notation, draw the following graphs:

Graph 2

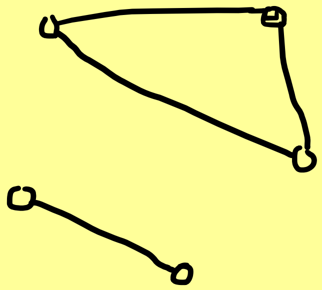
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$



Minimum degree : 1
Maximum degree: 4
No. of components : 1
Size of largest clique : 4

Graph 3

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



Minimum degree: 1
Maximum degree: 2
No. of components : 2
Size of largest clique: 3

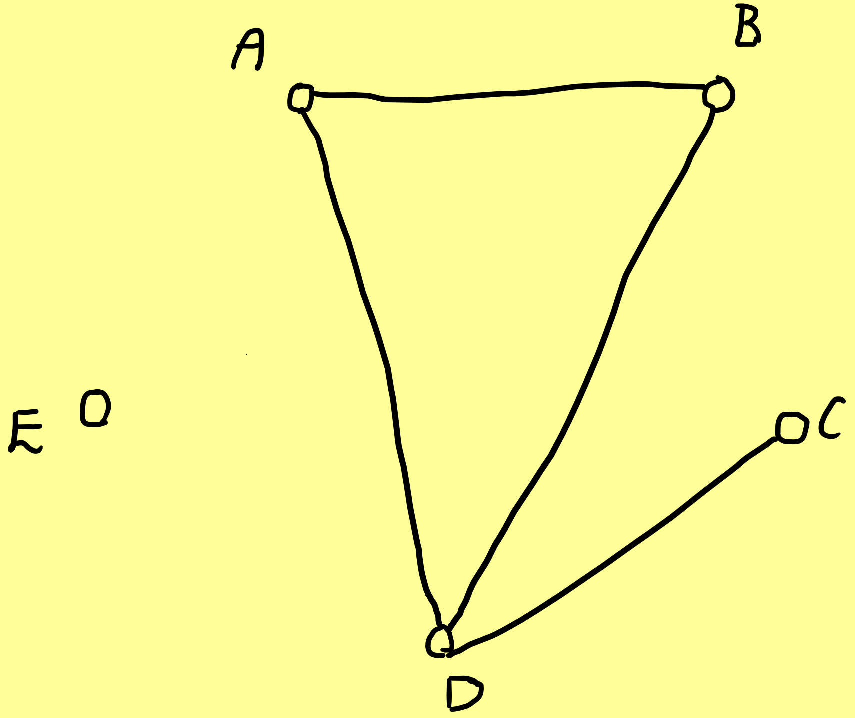
a) What is the average and minimum and maximum degree of each graph?
b) How many connected components are there?
c) What's the size of the largest clique?

Graph Theory Questions

	A	B	C	D	E	
A	0	1	0	1	0	→ 2 +
B	1	0	0	1	0	→ 2 +
C	0	0	0	1	0	→ 1 +
D	1	1	1	0	0	→ 3 +
E	0	0	0	0	0	→ 0

$2 + 2 + 1 + 3 + 0 = 8$

$\frac{1}{2} \times 8 = 4$ edges



Graph Theory Questions

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3) Draw all possible graphs with 3 vertices.

4) Can you draw a graph with:

a) 5 vertices with degrees 3,2,2,1,0
b) 5 vertices of degrees 4,4,3,3,2
c) 5 vertices with total degree 14
d) 6 vertices with degrees 4,2,2,2,2,1
e) 6 vertices with degrees 4,4,4,4,2,2
f) 6 vertices with degrees 3,3,2,2,1,1
g) 8 vertices, with at least 2 vertices of degree 3 and maximum degree 7 and minimum degree 2

For the graphs in part 4, identify the size and number of:

- Cliques
- Components
- Isolated vertices

Are there more than one way of drawing these graphs?

Graph Theory Questions

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5) Can you draw a graph with two connected components:
- One component is a clique with 4 vertices
- Another component has 5 vertices and total degree 10.

6) What is the maximum possible degree of a vertex in a graph with 6 vertices?
  - What about with n vertices?
  - What about if the graph is disconnected.

7) What do you notice about:
- The total degree of the graph?
- The number of odd and even degree vertices in a graph?

8) Does there exist a graph with two connected components:

  a) One component is a clique with 4 vertices

  b) Another component has 5 vertices and total degree 10?

Why or why not?

Extension Question:
Prove that every graph with 2 or more vertices must have at least two vertices with the same degree.

Graph Theory Definitions

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Formal Graph Notation

Let **G**=(V,E). **G** is the graph. V are the vertices. E are the edges.

Degree: The degree of a vertex is the number of connections coming from the vertex.
Notation: deg(B) for degree of vertex B.
For graph G, deg(G) is the sum of the degrees of all vertices in the graph.

Δ(G)= Maximum degree of a graph, G.
δ(G)= Minimum degree of a graph, G.

Isolated vertex: A vertex which is connected to no other vertices (i.e. a vertex with degree zero).

Connected: A graph is connected if for each vertex you can get to any other vertex by travelling along a series of edges.

Components: A component is a connected graph or subgraph.

Clique: A subset of vertices where each pair of vertices are connected by an edge.