Welcome back to Extensions!
Here are some things to do whilst we wait for everyone to join us…

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1. **Square puzzles**

   Suppose you have a 30×7 rectangle, and want to cover it with squares. The squares can't overlap, and you want to use as few squares as possible, but the squares could be of different sizes (e.g. some 1×1, some 2×2, some 3×3 and so on).

   What's the most efficient way of doing so? How did you arrive at this conclusion? What about rectangles of other dimensions (assume both sides have integer lengths)?

2. Find as many solutions to $x^2 - 2y^2 = 1$ as you can, where x and y are both integers. Then do the same with the equation $x^2 - 2y^2 = -1$.

Our shared expectations

I will treat all other participants, students and teachers alike, with respect and with compassion.

I will, both during these sessions and afterwards, treat all participants equally regardless of their background or identity.

I will not, either during these sessions or afterwards, bully, harass, intimidate or discriminate against any participant in these sessions.

I will not record or capture any video or images (e.g. screenshots) during these sessions.

I will follow all instructions given to me to the best that I can.

I will engage in these sessions with tenacity and resilience. I will always 'have a go'.

From the intro slide

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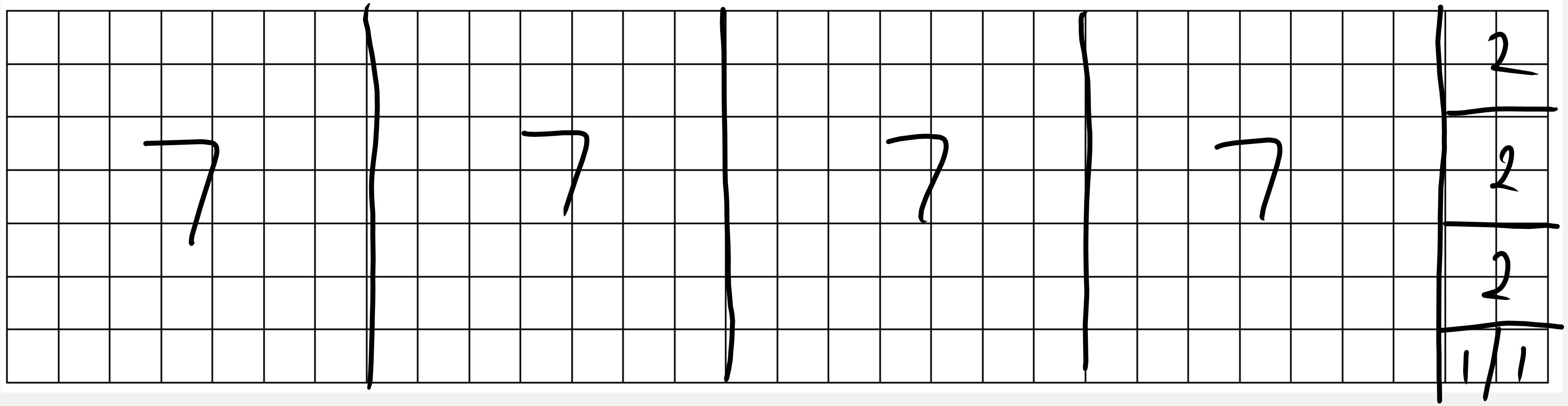
1. Square puzzles

30×7: four 7×7 squares, then three 2×2 squares, then two 1×1 squares.



"Try to use the biggest square you can in the remaining space"

From the intro slide

2. Solutions to $x^2 - 2y^2 = 1$:

$x = \pm 1, y = 0$

$x = \pm 3, y = \pm 2$

$\underline{x = 17, y = 12}$

For $x^2 - 2y^2 = -1$:

$x = \pm 1, y = \pm 1$

$\underline{x = \pm 7, y = \pm 5}$

$\underline{x = 41, y = 29}$

Intro to continued fractions

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Here's an interesting way of representing the fraction 30/7:

$$\frac{30}{7} = 4 + \boxed{\frac{2}{7}}$$

integer + (something between 0 and 1)

take reciprocal

$$= 4 + \frac{1}{\left(\frac{7}{2}\right)}$$

$$= 4 + \frac{1}{3 + \boxed{\frac{1}{2}}} = [4; 3, 2]$$

STOP once the $\boxed{}$ is of the form $\frac{1}{n}$

Intro to continued fractions

$$\frac{13}{8} = 1 + \frac{5}{8} \qquad [1; 1, 1, 1, 2]$$

$$= 1 + \frac{1}{\left(\frac{8}{5}\right)} = 1 + \frac{1}{1 + \frac{3}{5}}$$

$$= 1 + \frac{1}{1 + \frac{1}{\left(\frac{5}{3}\right)}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{2}{3}}}$$

$$= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\left(\frac{3}{2}\right)}}} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}$$

$\frac{1}{2}$ is of the form $\frac{1}{n}$, so we stop.

Continued Fractions

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If a_0 is an integer, and if $a_1, a_2, \dots, a_n$ are positive integers, then the expression:

Normal fraction $= \frac{a}{b}$

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

Either:
-> stop once you reach a_n
-> or, could go on infinitely

is called a **continued fraction**. The expression is also denoted by the shorthand notation:

$$[a_0; a_1, a_2, a_3, \dots, a_n]$$

Continued Fractions example

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$[1\,;2,2]$ can be rewritten in the form a/b as follows:

$$[1;2,2] = 1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{a}{b}$$

$$1 + \frac{1}{5/2} = 1 + \frac{2}{5} = \frac{7}{5}$$

Continued Fractions conversion questions

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Express the following as continued fractions:

1) $45/16$
2) $59/21$
3) $107/63$

Express the following continued fractions in the form a/b:

4) $[1;\ 2]$
5) $[1;\ 2, 2, 2]$
6) $[1;\ 2, 2, 2, 2]$

Continued Fractions conversion questions - answers

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Express the following as continued fractions:

1) 45/16

$$\frac{45}{16} = 2 + \frac{13}{16} = 2 + \frac{1}{\left(\frac{16}{13}\right)} = 2 + \frac{1}{1 + \frac{3}{13}} = 2 + \frac{1}{1 + \frac{1}{\left(\frac{13}{3}\right)}}$$

$$= 2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{3}}} = [2; 1, 4, 3]$$

similar as continued fractions, but look different as "normal" fractions.

2) 59/21

$\to [2; 1, 4, 4]$

3) 107/63

$\to [1; 1, 2, 3, 6]$

Continued Fractions conversion questions - answers

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Express the following continued fractions in the form a/b:

4) $[1;\ 2]$ $\rightarrow 1+\frac{1}{2} = 3/2$

5) $[1;\ 2, 2, 2]$

$$\hookrightarrow 1+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}} = 1+\cfrac{1}{2+\frac{1}{(5/2)}} = 1+\cfrac{1}{2+\frac{2}{5}} = 1+\frac{1}{(12/5)}$$

$$= 1+\frac{5}{12}$$

$$= 17/12$$

6) $[1;\ 2, 2, 2, 2]$

$\hookrightarrow \frac{41}{29}$

Continued Fraction representation of $\sqrt{2}$

Firstly, note that $\sqrt{2} = 1 + (\sqrt{2} - 1)$, and that $(\sqrt{2} - 1)$ is between 0 and 1.

$$\sqrt{2} = 1 + (\sqrt{2} - 1)$$

$$= 1 + \frac{1}{\left(\frac{1}{\sqrt{2} - 1}\right)}$$

$$= 1 + \frac{1}{(\sqrt{2} + 1)} \leftarrow 2.4\ldots\ldots$$

$$= 1 + \frac{1}{2 + (\sqrt{2} - 1)}$$

But: $\dfrac{1}{\sqrt{2} - 1} = \dfrac{1(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)}$

$$= \frac{\sqrt{2} + 1}{2 - 1}$$

$$= \sqrt{2} + 1$$

Continued Fraction representation of $\sqrt{2}$

$$\sqrt{2} = 1 + \cfrac{1}{2 + (\sqrt{2} - 1)}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{\left(\cfrac{1}{\sqrt{2} - 1}\right)}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{(\sqrt{2} + 1)}}$$

$$= 1 + \cfrac{1}{2 + \cfrac{1}{2 + (\sqrt{2} - 1)}}$$

$$[1; \overline{2}]$$

$$\parallel$$

$$= [1; 2, 2, 2, \ldots]$$

Continued Fraction representation of $\sqrt{n}$ questions

Show that:

1) $\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \dots]$ (i.e. "1, 2" repeating)
2) $\sqrt{5} = [2; 4, 4, 4, \dots]$ (i.e. 4 repeating)
3) $\sqrt{6} = [2; 2, 4, 2, 4, 2, 4, \dots]$ (i.e. "2, 4" repeating)
4) $\sqrt{7} = [2; 1, 1, 1, 4, 1, 1, 1, 4, \dots]$ (i.e. "1, 1, 1, 4" repeating)

pick one.

5) What is the exact value of $[1; 1, 1, 1, \dots]$ (i.e. 1 repeating)?
Hint: let $\alpha = [1; 1, 1, 1, \dots]$. Then $\alpha - 1 = \dots$

6) Investigate the value of $[1; 1, 1, \dots, 1]$ (where there are n 1s after the ;).
7) Investigate the value of $[1; 2, 2, \dots]$ (i.e. the continued fraction representation of $\sqrt{2}$) where there are e.g. four/five/six or more 2s after the ;).

Continued Fraction representation of $\sqrt{n}$ questions – some answers

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Show that:

1) $\sqrt{3} = [1; 1, 2, 1, 2, 1, 2, \dots]$ (i.e. "1, 2" repeating)

$$\sqrt{3} = 1 + (\sqrt{3} - 1)$$

$$= 1 + \frac{1}{\left(\frac{1}{\sqrt{3}-1}\right)}$$

$$\frac{1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\sqrt{3}+1}{2}$$

$$= 1.3\dots$$

$$= 1 + \frac{1}{1 + \left(\frac{\sqrt{3}-1}{2}\right)}$$

$$1 + \frac{1}{1 + \frac{1}{\left(\frac{2}{\sqrt{3}-1}\right)}}$$

Continued Fraction representation of $\sqrt{n}$ questions – some answers

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$$\sqrt{3} = 1 + \cfrac{1}{1 + \cfrac{1}{\left(\frac{2}{\sqrt{3}-1}\right)}}$$

$$\frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{\cancel{2}(\sqrt{3}+1)}{\cancel{2}}$$

$$= \sqrt{3}+1$$

$$= 2.7\ldots$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{2 + (\sqrt{3}-1)}}$$ ← same as earlier....

$$= [1; 1, 2, 1, 2, 1, 2, \ldots] = [1; \overline{1, 2}]$$

Continued Fraction representation of $\sqrt{n}$ questions – some answers

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5) What is the exact value of $[1; 1, 1, 1, \dots]$ (i.e. 1 repeating)?

Hint: let $\alpha = [1; 1, 1, 1, \dots]$. Then $\alpha - 1 = \dots$

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 \dots}}} \qquad \alpha - 1 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 \dots}}} = \frac{1}{\alpha}$$

So α satisfies;

$$\alpha - 1 = \frac{1}{\alpha}$$

$$\alpha^2 - \alpha = 1$$

$$\alpha^2 - \alpha - 1 = 0 \Rightarrow \alpha = \frac{1 + \sqrt{5}}{2}$$

Continued Fraction representation of $\sqrt{n}$ questions – some answers

6) Investigate the value of $[1; 1, 1, \dots, 1]$ (where there are n 1s after the ;).

$\frac{3}{2}$ $\frac{5}{3}$ $\frac{8}{5}$ $\frac{13}{8}$ etc. "ratio of consecutive fibonacci numbers".

7) Investigate the value of $[1; 2, 2, \dots]$ (i.e. the continued fraction representation of $\sqrt{2}$) where there are e.g. four/five/six or more 2s after the ;).

$[1;2] = \frac{3}{2}$ $17/12$ $99/70$ $\to x^2 - 2y^2 = 1$

$[1;2,2] = \frac{7}{5}$ $41/29$ $239/169$ $\to x^2 - 2y^2 = -1$

Diophantine equations

A Diophantine equation is an equation in one or more variables, where we only care about *integer solutions*. In general they are hard/impossible to solve.

However, for certain special cases, there are techniques to solve them.

Linear Diophantine equations: e.g. $ax = b$ or $ax + by = c$ $(a, b, c$ integers)

Euclid's algorithm.
(extended version)

Diophantine equations and Pell's equation

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Quadratic Diophantine equations: e.g. $ax^2 + bx + cy^2 + dy + exy + f = 0$ (*a, b, c, d, e, f* integers)

Diophantine equations and Pell's equation

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Pell's equations: $x^2 - Dy^2 = 1$ (D a positive integer that isn't a square).

Look at the continued fraction of $\sqrt{D}$ and <u>truncate</u> it.

eg. $D = 2$ $\quad (x^2 - 2y^2 = 1)$ $\quad \sqrt{2} = [1; \overline{2}]$

"every other approximation"

$[1; 2] = \frac{3}{2} \to x = 3,\ y = 2$

$[1; 2, 2, 2] = \frac{17}{12} \to$

Continued Fraction representation of π

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Firstly, we know that $\pi = 3 + (some\ value\ between\ 0\ and\ 1)$.

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{299 \ldots}}}}$$

$$3 + \frac{1}{7} = \frac{22}{7}$$

$$3 + \cfrac{1}{7 + \cfrac{1}{15}} = \frac{333}{106}$$

$$3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}} = 355/113$$

accurate to 6 d.p.
Fantastic approximation to π

Continued Fraction representation of π

Some things to think about if you want to explore further…

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Pell's equation

Use the continued fraction representations of $\sqrt{3}$ (or $\sqrt{5}$, or $\sqrt{6}$, or $\sqrt{7}$) to obtain some more integer solutions to $x^2 - Dy^2 = 1$, in the case $D = 3,5,6$ or 7.

Continued Fractions and Euclid's Algorithm

Read up more on Continued Fractions, and how ideas such as Euclid's algorithm can help with calculations.

Extra slide

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CHANGE TEXT HERE