Welcome to Extensions!
Here are some things to do whilst we wait for everyone to join us…

KING'S MATHS SCHOOL

1. Grab some pen and paper, and your calculator.

2. Register for a free account at www.whiteboard.chat/ (an online whiteboard we'll use for these sessions). Click "Start Drawing" then "Manage Boards".

3. Introduce yourself in the Zoom chat! You could type in a sentence about why you like maths, for example.

4. Think about how you might tackle the following questions:
   a) What's the last digit of the number 17^{17}?
   b) What's the remainder when you divide 17^{17} by 27?

Welcome to Extensions!

KING'S MATHS SCHOOL

1. **Mondays 5:00 – 6:30pm** (no session currently planned for 14th Feb, final session on 28th March), same Zoom link each week.

2. Maths topics you might not get to see at GCSE (or even A Level!).
   So, something a bit different and hopefully harder than what you might be used to.

3. You don't have to do any work outside of these sessions – but it doesn't hurt to stretch yourself!

Our shared expectations

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I will treat all other participants, students and teachers alike, with respect and with compassion.

I will, both during these sessions and afterwards, treat all participants equally regardless of their background or identity.

I will not, either during these sessions or afterwards, bully, harass, intimidate or discriminate against any participant in these sessions.

I will not record or capture any video or images (e.g. screenshots) during these sessions.

I will follow all instructions given to me to the best that I can.

I will engage in these sessions with tenacity and resilience. I will always 'have a go'.

From the warm up…

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a) What's the last digit of the number 17^{17}? 7 ?

Some kind of repeating pattern ?

b) What's the remainder when you divide 17^{17} by 27?

Not so obvious ???

Modular arithmetic introduction

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If n is a positive integer, and x and y are any integers, then we write

$$\boxed{x \equiv y \ (mod\ n)}$$

to mean any of the following equivalent statements:

1) x and y give the same remainder when you divide them by n
2) x is y more than a multiple of n
3) $x = y + kn$ for some integer k
4) $(x - y)$ is divisible by n
5) n is a factor of $(x - y)$

Modular arithmetic introduction #1

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$$x \equiv y \ (mod\ n)$$

1) x and y give the same remainder when you divide them by n

eg. $35 \equiv 17 \pmod 6$ because 35 and 17 both give remainder 5 when you divide them by 6.

2) x is y more than a multiple of n

35 is 17 more than 18 (18 is a multiple of 6)

Modular arithmetic introduction #2

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$$x \equiv y \ (mod\ n)$$

3) $x = y + kn$ for some integer k

$35 = 17 + 3(6)$

4) $(x - y)$ is divisible by n

5) n is a factor of $(x - y)$

$35 - 17 = 18$, and 18 is divisible by 6

(6 is a factor of 18)

Modular arithmetic short questions

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True or False?

a) $10 \equiv 2\ (mod\ 4)$

b) $17 \equiv 5\ (mod\ 3)$

c) $24 \equiv -10\ (mod\ 5)$ → False

d) $-3 \equiv -13\ (mod\ 5)$

e) $2 \equiv 10\ (mod\ 4)$

f) $57 \equiv 25\ (mod\ 4)$

g) $59 \equiv 35\ (mod\ 4)$

h) $114 \equiv 250\ (mod\ 4)$ → "114 is congruent to 250 modulo 4"

All True, except for

Remember that:

$x \equiv y\ (mod\ n)$

is equivalent to any of the following statements:

1) x and y give the same remainder when you divide them by n
2) x is y more than a multiple of n
3) $x = y + kn$ for some integer k
4) $(x - y)$ is divisible by n
5) n is a factor of $(x - y)$

"x is congruent to y modulo n"

Modular arithmetic short questions - answers

KING'S MATHS SCHOOL

True or False?

a) $10 \equiv 2 \ (mod\ 4)$

b) $17 \equiv 5 \ (mod\ 3)$

c) $24 \equiv -10 \ (mod\ 5)$

d) $-3 \equiv -13 \ (mod\ 5)$

e) $2 \equiv 10 \ (mod\ 4)$

f) $57 \equiv 25 \ (mod\ 4)$

g) $59 \equiv 35 \ (mod\ 4)$

h) $114 \equiv 250 \ (mod\ 4)$

a) and e)

e), f) and g)

~~f), g)~~

e), f) and h)

Modular arithmetic algebraic laws #1

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a) and e) hint that if $a \equiv b \ (mod\ n)$ then it's also true that $b \equiv a \ (mod\ n)$.

Mini proof:

4 $a - b$ is a multiple of n.

Therefore, $b - a$ is a multiple of n.

$b - a = -(a - b)$

This means $b \equiv a \ (mod\ n)$

Modular arithmetic algebraic laws #2

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e), f) and g) hint that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$ then:

$$a + c \equiv b + d \ (mod\ n)$$

Mini proof:

3 $a = b + k_1 n$

$c = d + k_2 n$ } for some integers k_1 and k_2

$35 \equiv 17 \ (mod\ 6)$

$35 = 17 + 3(6)$

$12 \equiv 6 \ (mod\ 6)$

$12 = 6 + 1(6)$

Add: $a + c = b + d + \underbrace{(k_1 + k_2)}n$ → call this k, integer

$a + c = b + d + kn$

So: $(a + c) \equiv (b + d) \ (mod\ n)$

Modular arithmetic algebraic laws #3

e), f) and h) hint that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$ then:

$$ac \equiv bd \ (mod\ n)$$

Mini proof:

$$\left.\begin{aligned} a &= b + k_1 n \\ c &= d + k_2 n \end{aligned}\right\} \text{ for some integers } k_1 \text{ and } k_2$$

$$ac = (b + k_1 n)(d + k_2 n) = bd + bk_2 n + dk_1 n + k_1 k_2 n^2$$

$$ac = bd + \underbrace{(bk_2 + dk_1 + k_1 k_2 n)}_{\text{call this } k, \text{ an integer}} n$$

So: $ac \equiv bd \ (mod\ n)$

Modular arithmetic more questions

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1) Explain why, if $a \equiv b \ (mod\ n)$ then $a^m \equiv b^m \ (mod\ n)$ for any positive integer $m \geq 2$.
2) Justify the following statement: $x \equiv x \ (mod\ n)$.
3) $60 \equiv x \ (mod\ 7)$ for some integer x where $0 \leq x < 7$. Find x.
4) $(19 \times 20) + 21 \equiv x \ (mod\ 16)$ for some integer x, where $0 \leq x < 16$. Find x.
5) Any integer x is congruent to **exactly one of** $0,1,2, \dots, n-1$ modulo n. Justify this statement.

Modular arithmetic more questions - answers

KING'S MATHS SCHOOL

1) Explain why, if $a \equiv b\ (mod\ n)$ then $a^m \equiv b^m\ (mod\ n)$ for any positive integer $m \geq 2$.

$$\frac{\begin{array}{l} a \equiv b\ (mod\ n) \\ a \equiv b\ (mod\ n) \end{array}}{a^2 \equiv b^2\ (mod\ n)} \qquad \frac{\begin{array}{l} a^2 \equiv b^2\ (mod\ n) \\ a \equiv b\ (mod\ n) \end{array}}{a^3 \equiv b^3\ (mod\ n)} \qquad \text{etc....}$$

2) Justify the following statement: $x \equiv x\ (mod\ n)$.

* x and x have the same remainder when you divide them by n.

** $x - x = 0$, which is divisible by n ($0 \div n = 0$, which is a whole number)

Modular arithmetic more questions - answers

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3) $60 \equiv x \; (mod\ 7)$ for some integer x where $0 \leq x < 7$. Find x.

$x = 4$

4) $(19 \times 20) + 21 \equiv x \; (mod\ 16)$ for some integer x, where $0 \leq x < 16$. Find x.

$19 \equiv 3 \; (mod\ 16)$

$20 \equiv 4 \; (mod\ 16)$

$21 \equiv 5 \; (mod\ 16)$

$(19 \times 20) + 21 \equiv (3 \times 4) + 5 = 17$

$17 \equiv 1 \; (mod\ 16)$

So $\underline{x = 1}$

Modular arithmetic more questions - answers

KING'S MATHS SCHOOL

5) Any integer x is congruent to **exactly one of** $0,1,2,\ldots,n-1$ modulo n. Justify this statement.

eg. $n = 7$

0	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30	31	32	33	34

Every integer in the 0 column is congruent to 0 mod 7.

Modular arithmetic harder worked examples #1

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Let's find the remainder when $\underline{13^{71}}$ is divided by $\underline{12}$.

Find x: $13^{71} \equiv x \pmod{12}$ and $0 \leq x < 12$

$13 \equiv \, ? \pmod{12}$

Well, $13 \equiv 1 \pmod{12}$

So: $13^{71} \equiv 1^{71} \pmod{12}$

$13^{71} \equiv 1 \pmod{12}$

$\boxed{\text{Remainder} = 1}$

Modular arithmetic harder worked examples #2

KING'S MATHS SCHOOL

Let's find the remainder when 3^{1989} is divided by 7.

What multiple of 6 is close to 1989?

$3^2 \equiv 2 \pmod 7$

$3^3 \equiv 6 \pmod 7$

$18 \equiv 4 \pmod 7$

$3^4 \equiv 4 \pmod 7$

$12 \equiv 5 \pmod 7$

$3^5 \equiv 5 \pmod 7$

$3^6 \equiv 1 \pmod 7$ → useful: $3^{6m} \equiv 1 \pmod 7$

$3^{1986} = (3^6)^{331} \equiv 1 \pmod 7$

$3^3 3^{1986} \equiv 6 \times 1 \pmod 7$

$3^{1989} \equiv 6 \pmod 7$

Modular arithmetic yet more questions

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1) What is the remainder when 9^{100} is divided by 8?
2) What is the remainder when 4^{2022} is divided by 5?
(Hint: $4 \equiv -1 \ (mod\ 5)$)
3) What is the remainder when 17^{17} is divided by 10?
(Hint: $17 \equiv 7 \ (mod\ 10)$, and $7^2 \equiv -1 \ (mod\ 10)$).
4) Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Modular arithmetic yet more questions - answers

1) What is the remainder when 9^{100} is divided by 8?

$9 \equiv 1 \pmod 8$, so $9^{100} \equiv 1^{100} = 1 \pmod 8$ → remainder = 1

2) What is the remainder when 4^{2022} is divided by 5?

(Hint: $4 \equiv -1\ (mod\ 5)$)

$4 \equiv (-1) \pmod 5$

$4^2 \equiv 1 \pmod 5$

$(4^2)^{1011} \equiv 1 \pmod 5$

$4^{2022} \equiv 1 \pmod 5$

remainder = 1

Modular arithmetic yet more questions - answers

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3) What is the remainder when 17^{17} is divided by 10?
(Hint: $17 \equiv 7 \ (mod\ 10)$, and $7^2 \equiv -1 \ (mod\ 10)$).

$17^4 \equiv 7^4 \ (\mathrm{mod}\ 10)$

$17^4 \equiv (7^2)^2 \ (\mathrm{mod}\ 10)$

$17^4 \equiv (-1)^2 \ (\mathrm{mod}\ 10)$

$17^4 \equiv 1 \ (\mathrm{mod}\ 10)$

$(17^4)^4 \equiv 1^4 \ (\mathrm{mod}\ 10)$

$17^{16} \equiv 1 \ (\mathrm{mod}\ 10)$

$17^{17} \equiv 17 \ (\mathrm{mod}\ 10)$

$\equiv 7 \ (\mathrm{mod}\ 10)$

remainder = 7

Modular arithmetic yet more questions - answers

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4) Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Trickier; first show $2222^{5555} \equiv 5 \pmod 7$

$5555^{2222} \equiv 2 \pmod 7$

Adding gives the desired result!

Modular arithmetic even more questions for you to think about!

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1) What is the last digit of the number 3^{2011}? 2^{50}? 777^{777}?
2) Why is the number $n^5 + 4n$ always divisible by 5? (Hint: remember that n is congruent to either $0,1,2,3$ or 4 modulo 5, so you can just check every case).
3) What is the "tens" digit of $2013^2 - 2013$?
4) What's the remainder when 7^{243} is divided by 28.

5) Choose a prime p, and then pick any integer a which is not divisible by p. What is the remainder when you divide a^{p-1} by p? Experiment with different p and a values, and then make a conjecture!

Some things to think about before next week…

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Square puzzles

Suppose you have a 30×7 rectangle, and want to cover it with squares. The squares can't overlap, and you want to use as few squares as possible, but the squares could be of different sizes (e.g. some 1×1, some 2×2, some 3×3 and so on).

What's the most efficient way of doing so? How did you arrive at this conclusion?

What about rectangles of other dimensions (assume both sides have integer lengths)?

Some things to think about before next week…

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Example

Clearly you could cover a 3×8 rectangle with twenty-four 1×1 rectangles. But this is a lot of individual rectangles!

Here's an example of covering a 3×8 rectangle with squares:

1) Two 3×3 rectangles
2) One 2×2 rectangle
3) Two 1×1 rectangles.

For a 3×8 rectangle, you can't do any better!

Extra slide

CHANGE TEXT HERE