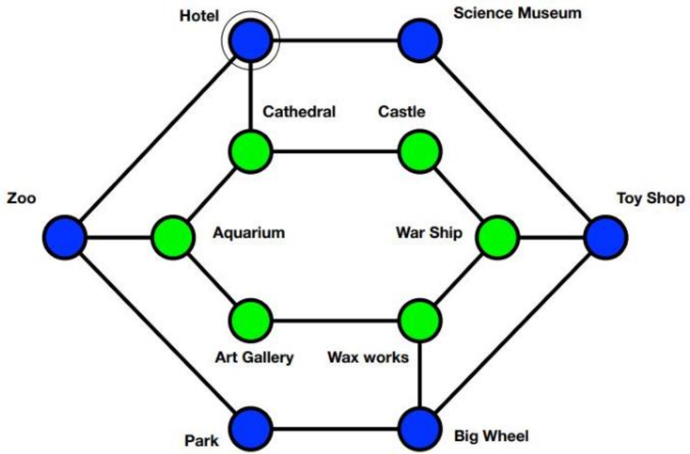
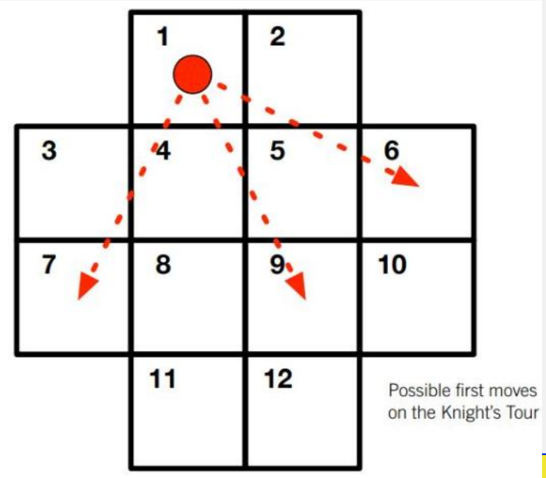
# Welcome back to Extensions!
# Here are some things to do whilst we wait for everyone to join us…

KING'S MATHS SCHOOL

1. You are a hotel tour guide. You must work out a route that starts at the Hotel, visits every attraction exactly once, and ends up back at the Hotel. Give one possible route!

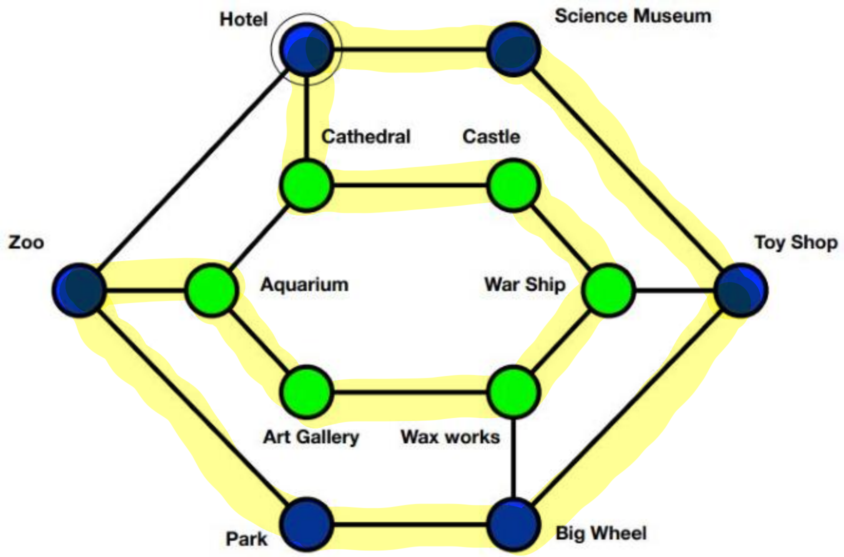



2. Consider the following unusual chessboard with only 12 squares. Find a "Knight's Tour", i.e. a sequence of moves that starts from square 1, visiting every square exactly once before returning to square 1.



# Intro problems

1. Hotel tour guide problem: here is one possible solution.

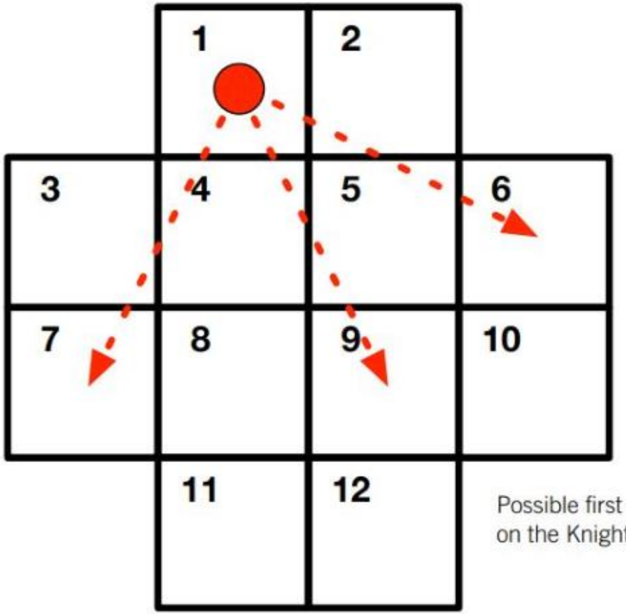



Hamiltonian cycle!
(visit every vertex
exactly once)

# Intro problems



2. Knight's Tour problem: here is one possible solution!




Possible first moves on the Knight's Tour

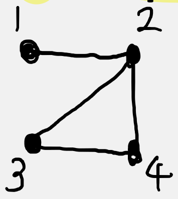
1 → 6 → 8 →
2 → 10 → 4 →
12 → 7 → 5 →
11 → 3 → 9 →
1

# Isomorphic graphs #1

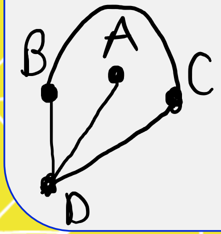
KING'S MATHS SCHOOL

Two graphs are said to be "isomorphic" if you can relabel/redraw one graph so that it "turns into" the other, with the new edges correctly matched up.

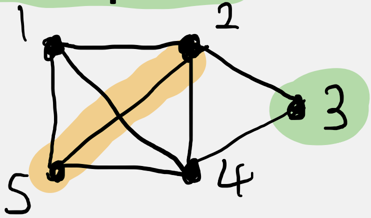
**Example 1:**

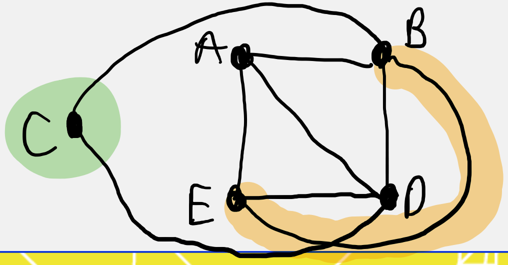



1 = A
4 = B
3 = C
2 = D




**Example 2:**



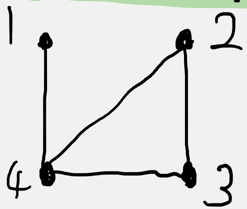

1 = A
2 = B
3 = C
4 = D
5 = E

# Isomorphic graphs #2

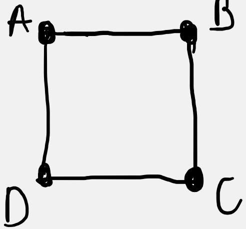
KING'S MATHS SCHOOL

Two graphs are said to be "isomorphic" if you can relabel/redraw one graph so that it "turns into" the other, with the new edges correctly matched up.

**Non-example 3:**




degrees = 1, 2, 2, 3




degrees = 2, 2, 2, 2

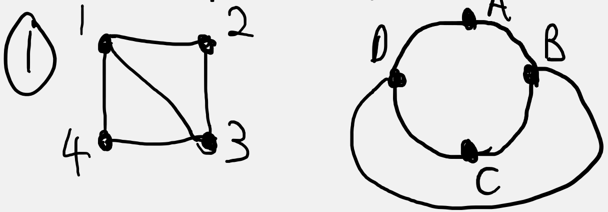
**Non-example 4:**

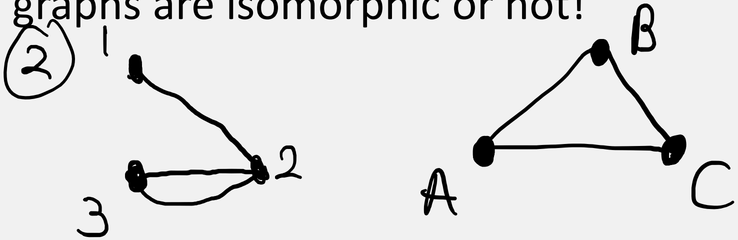
If the two graphs have different numbers of vertices, edges, collections of degrees, or cycles they are not isomorphic.

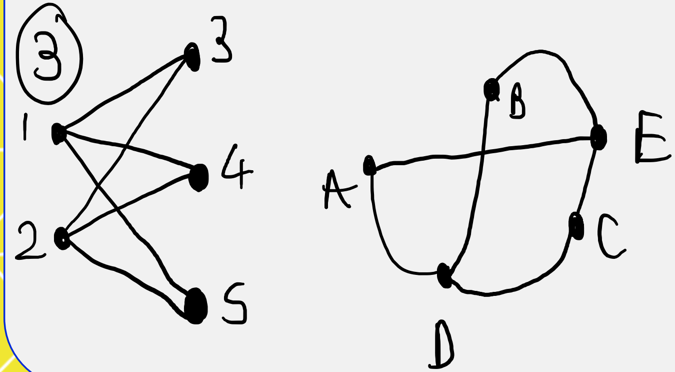
# Isomorphic graphs problems

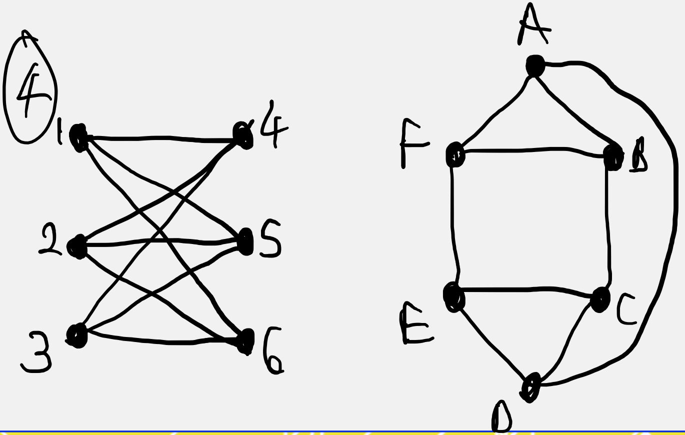
KING'S MATHS SCHOOL

In each problem, determine if the two graphs are isomorphic or not!



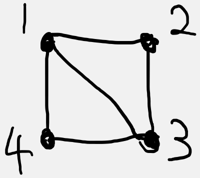



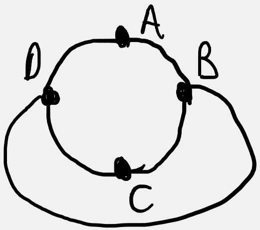



# Isomorphic graphs problems: rough solutions

1

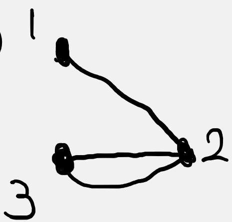


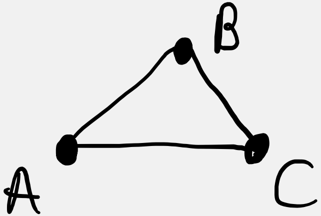




ISOMORPHIC!

eg. B, D = 1, 3
A, C = 2, 4

2



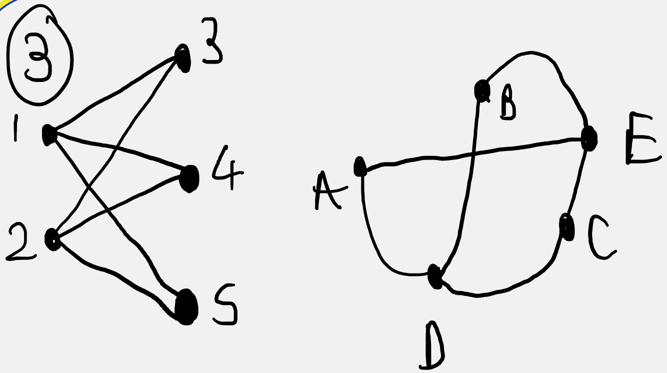

NOT ISOMORPHIC

degrees = 1, 2, 3

degrees = 2, 2, 2

# Isomorphic graphs problems: rough solutions

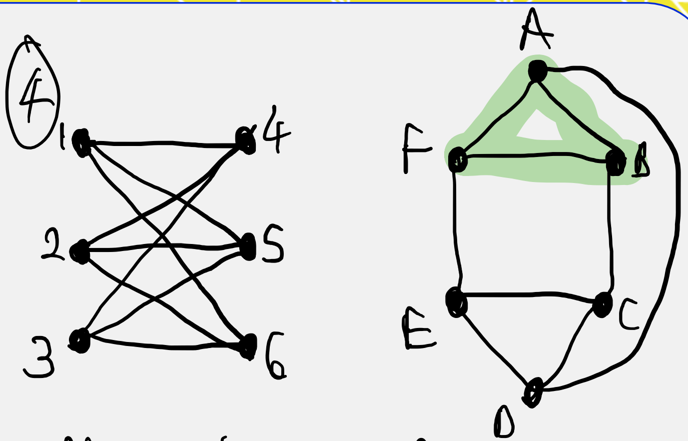
KING'S MATHS SCHOOL




ISOMORPHIC!

D, E = 1, 2

A, B, C = 3, 4, 5




NOT ISOMORPHIC

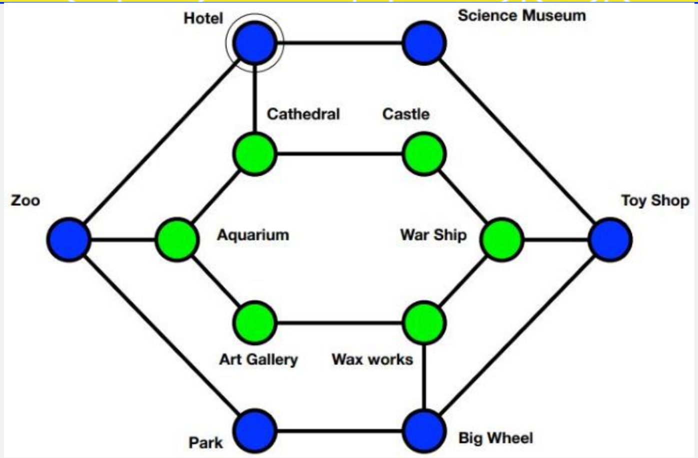
Both: 6 vertices, 9 edges, degree 3.

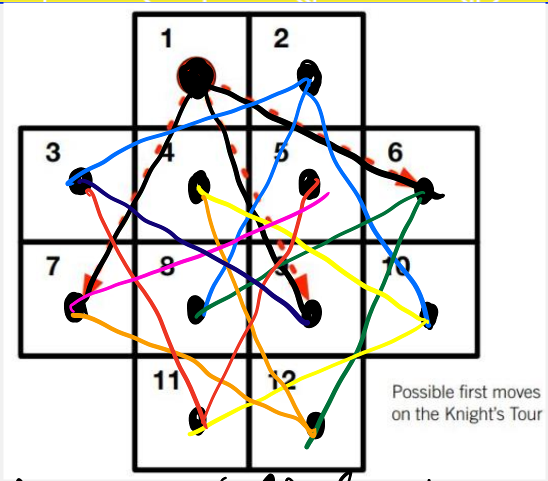
Right graph has cycles of length 3.

Left graph does not have cycles of length 3.

# Isomorphic graphs #3

KING'S MATHS SCHOOL

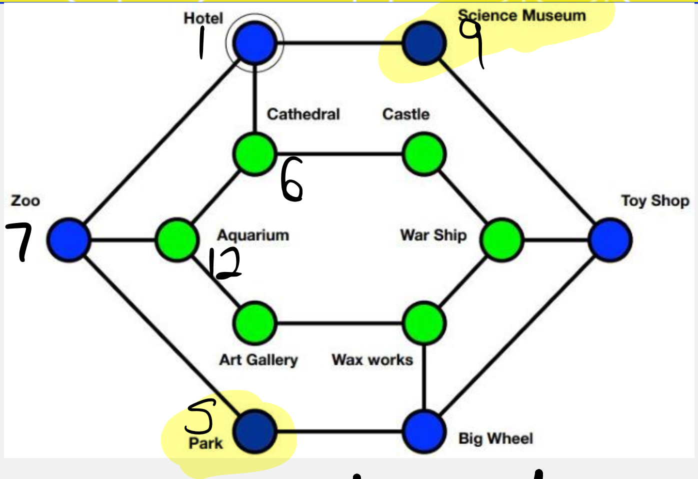


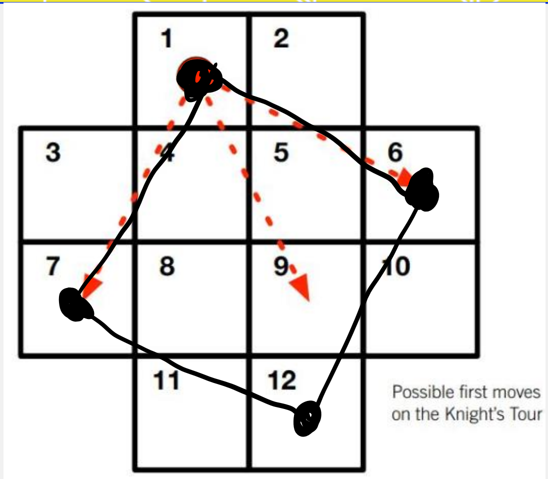




Turn the knight's tour into a graph.

# Isomorphic graphs #3

KING'S MATHS SCHOOL

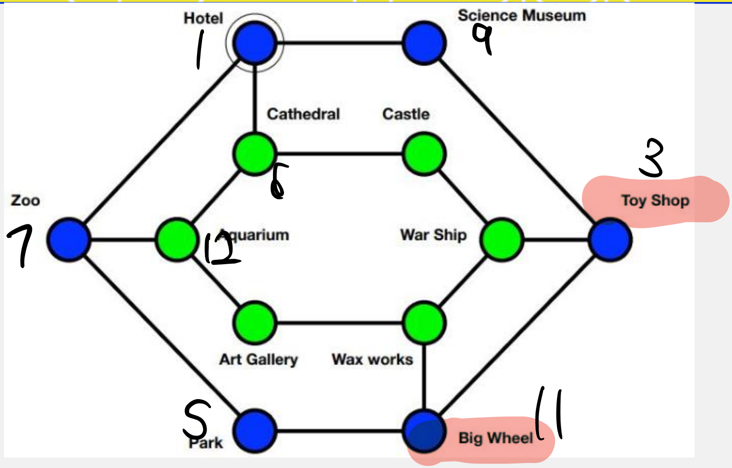


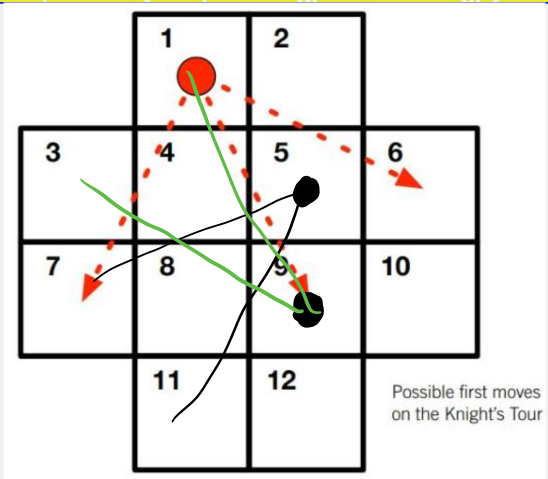




It turns out that the two graphs are isomorphic!

# Isomorphic graphs #3

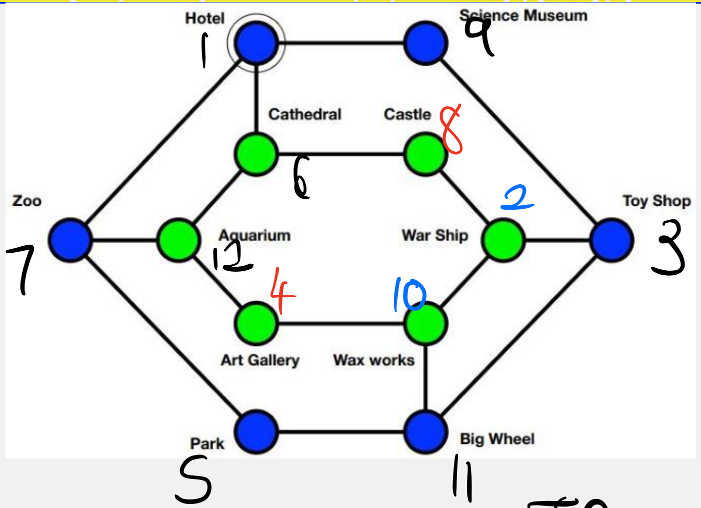
KING'S MATHS SCHOOL

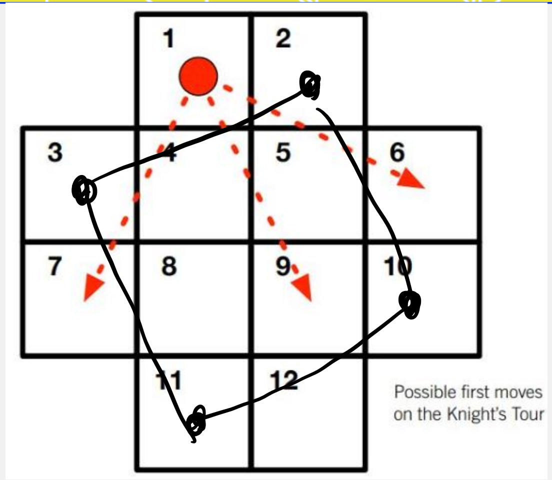


# Isomorphic graphs #3

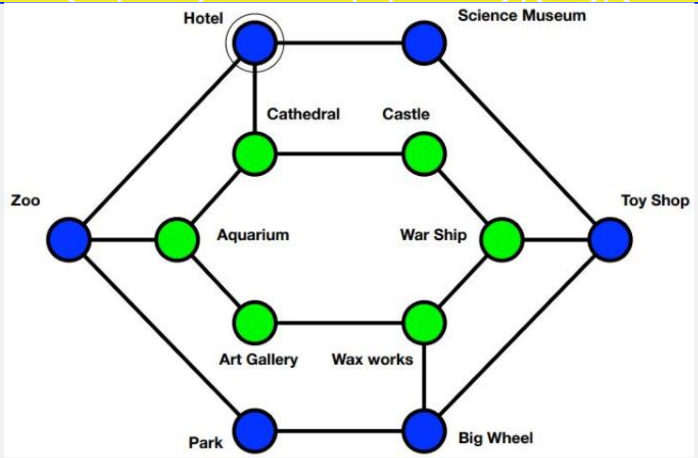
KING'S MATHS SCHOOL

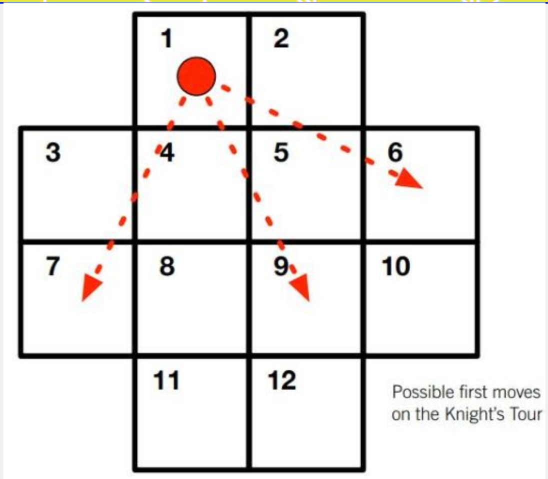



If possible try to draw a graph to represent the situation!

# Isomorphic graphs #3



# Problems that encourage you to draw a graph if you are stuck!

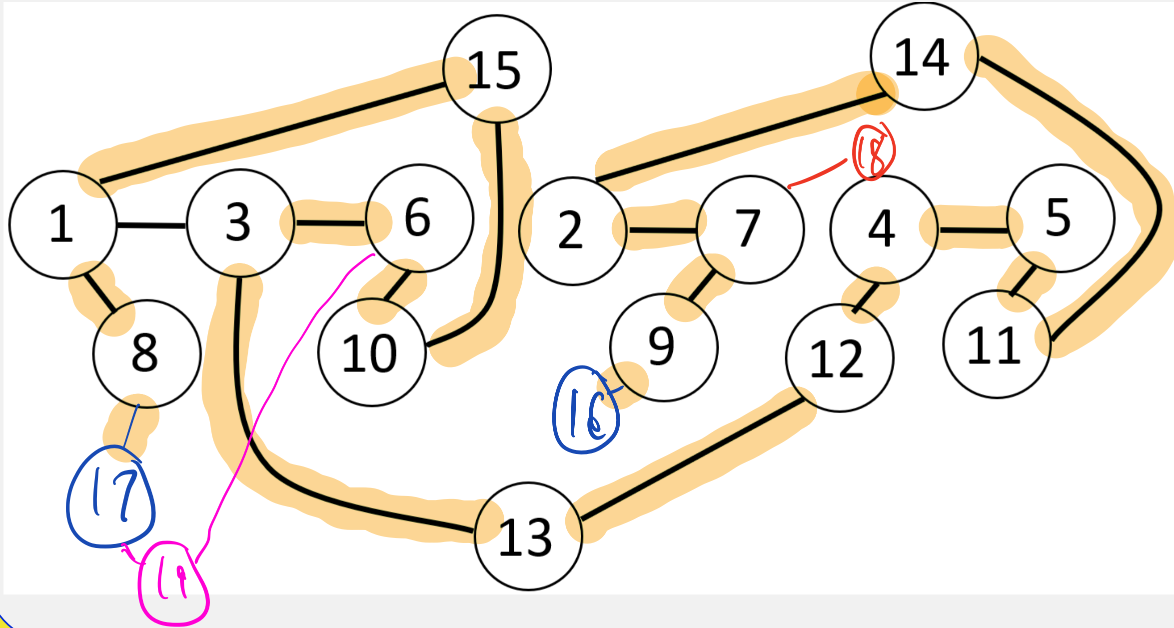
KING'S MATHS SCHOOL

1. Suppose you have the integers 1, 2, …, 15. Your challenge is to rearrange them into a new sequence of 16 numbers, so that any two consecutive terms add up to a square number. Note that you are not allowed to break up any of the two-digit numbers (10, 11, … , 15) and piece them together in a different way!
2. A fictitious bank gives all its customers very insecure PINs: each PIN is 2 digits long, and only uses the digits 0 or 1.
   a) Write down all 4 PINs
   b) The bank's ATMs behave weirdly. Instead of forcing customers to retype a whole new PIN if their previous attempt was wrong, it only checks if the last two digits match. For example, if you typed in 001 then this would be the same as having tried 00 and 01.
   Find a sequence of 5 digits that would "try" all two-digit PINs.

# Problems that encourage you to draw a graph if you are stuck!

KING'S MATHS SCHOOL

**1. Square number challenge.** Here is a graph that you could have drawn!




Hamiltonian path!

The problem is solvable if you go up to 16 or 17

(but not 18)

# Problems that encourage you to draw a graph if you are stuck!

KING'S MATHS SCHOOL

**2. PIN problem.**

00, 01, 10, 11

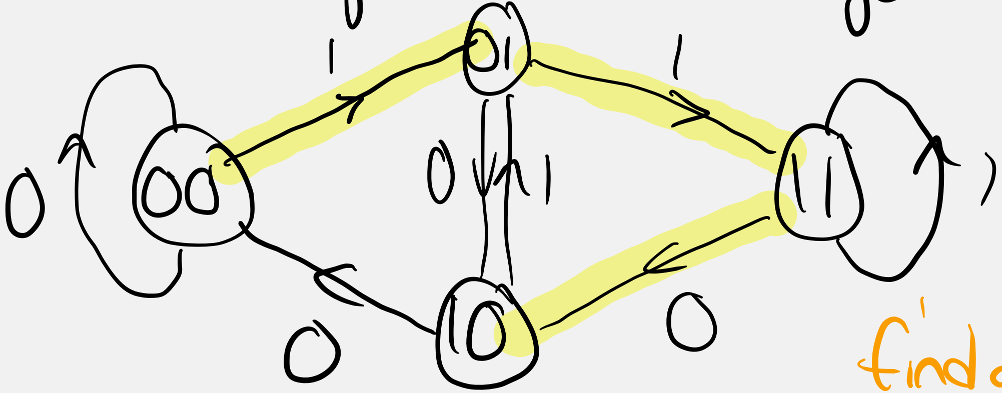
00110

00 01 11 10

"trial and error"

Vertices = 2 digit PINs

Edge = if you can "bridge" one PIN to the other.



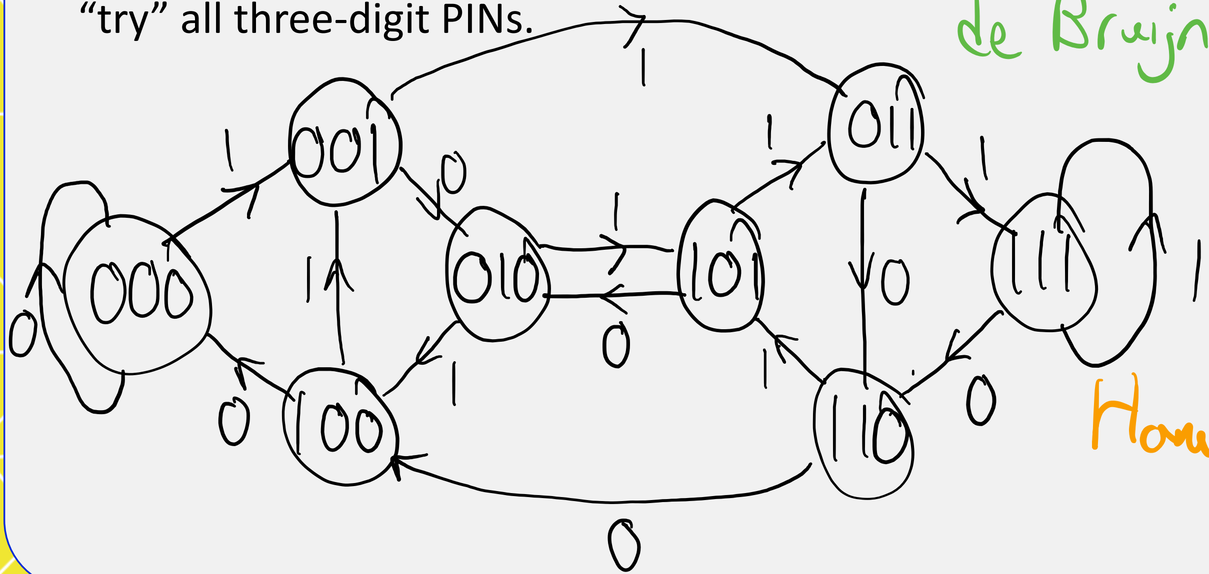

GOAL: find a Hamiltonian path

# Problems that encourage you to draw a graph if you are stuck!



**PIN problem continued.** In an effort to beef up security, the bank changes everyone's PIN to 3 digits long. Find a possible sequence of 10 digits that would "try" all three-digit PINs.

de Bruijn sequences

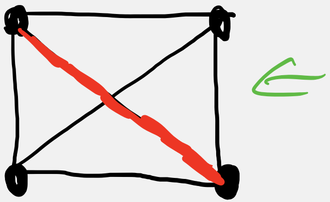


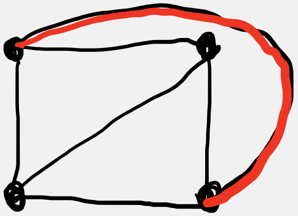

Find a Hamiltonian path!

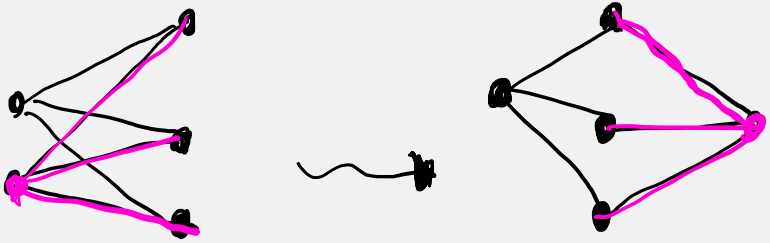
# Planar graphs #1

A planar graph is a graph that can be (possibly re-)drawn on a flat surface, so that the edges do not “cross over” each other.

**Several examples here.**



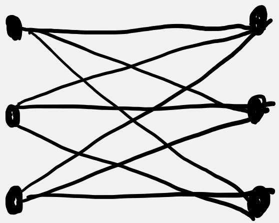




# Planar graphs #2

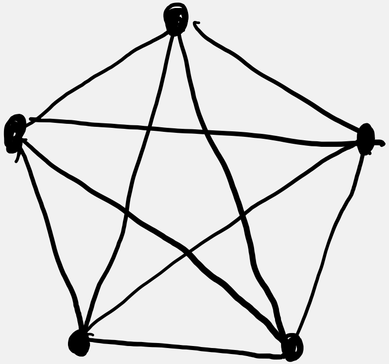
Here are two famous graphs which are *not* planar.

$K_{3,3}$



NOT planar

$K_5$



NOT planar!

# The platonic solids

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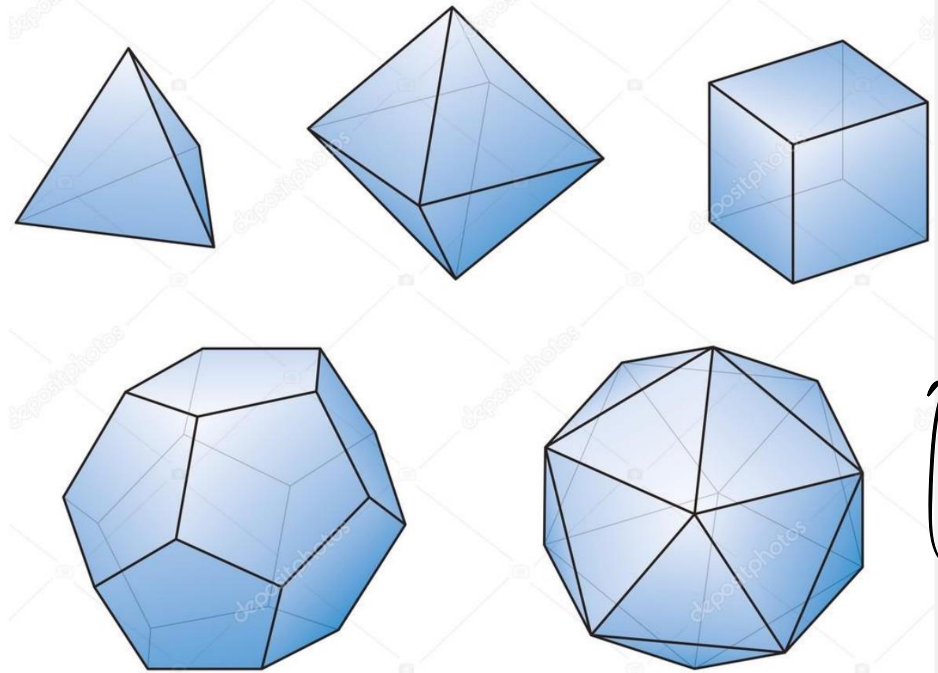


There are five platonic solids, i.e. 3D shapes made entirely of some number of regular polygons.

| Name | Vertices | Edges | Faces |
|---|---|---|---|
| Tetrahedron | 4 | 6 | 4 |
| Octahedron | 6 | 12 | 8 |
| Cube | 8 | 12 | 6 |
| Dodecahedron | 20 | 30 | 12 |
| Icosahedron | 12 | 30 | 20 |

$8 - 12 + 6 = 2$

# Planar graphs out of the platonic solids

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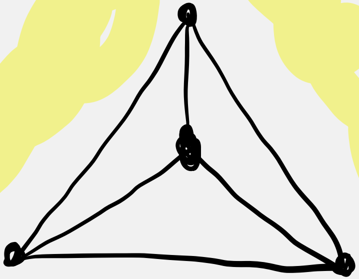
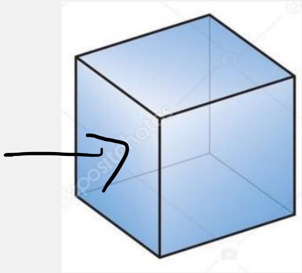
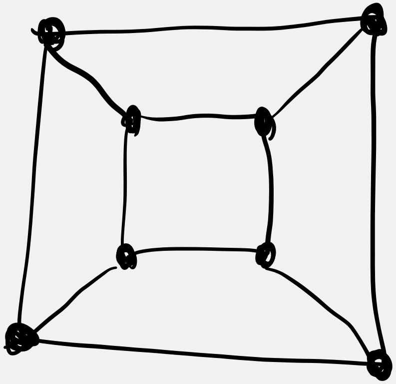
Imagine you press your face against one of the faces, and look "through" that face as though it were a window.

The platonic solid is then transformed into a planar graph!







# Planar graphs and Euler's formula

KING'S MATHS SCHOOL

The relationship $\boxed{V - E + F = 2}$ holds true for any connected planar graph ("connected" means you can get from any vertex to any other vertex by some sequence of edges).

Face = any region which is finite (bounded by edges) or the infinite region.

**Proof:**

One vertex and no edges

$V=1 \quad E=0 \quad F=1$

so $V-E+F=2$

# Proof of Euler's formula continued

KING'S MATHS SCHOOL

Build up your planar graph with the following two processes:

① Include a new vertex via an edge.

| | | | |
|---|---|---|---|
| Before: | $V$ | $E$ | $F$ |
| After: | $V+1$ | $E+1$ | $F$ |

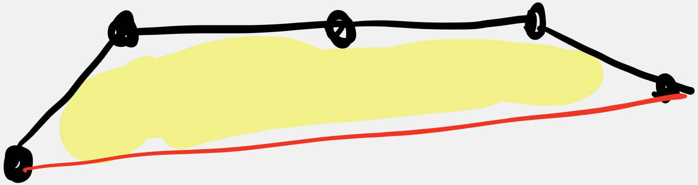
So $(V+1)-(E+1)+F = V-E+F$

$= 2$

# Proof of Euler’s formula continued

KING’S MATHS SCHOOL

(2) Introduce each remaining edge one at a time. Since the edge goes between two vertices that already have another sequence of edges between them:

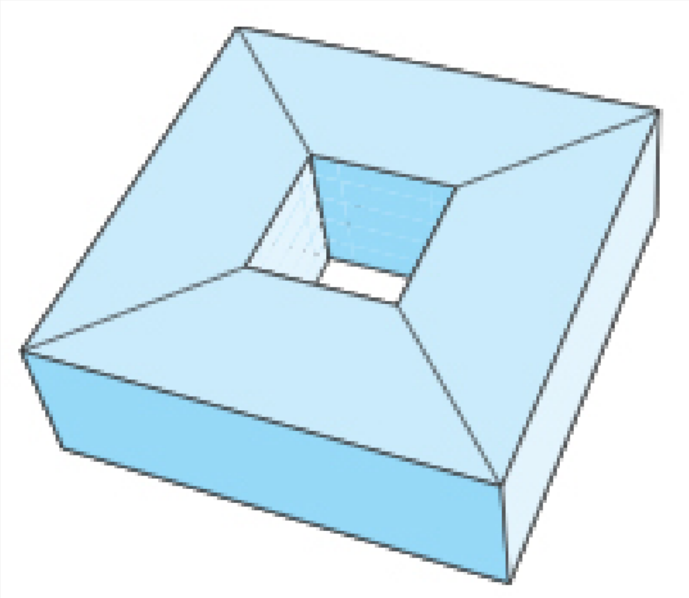


| | V | E | F |
|---|---|---|---|
| **Before:** | $V$ | $E$ | $F$ |
| **After:** | $V$ | $E+1$ | $F+1$ |

So $V-(E+1)+(F+1) = V-E+F = 2$

# Other 3D shapes…

KING'S MATHS SCHOOL

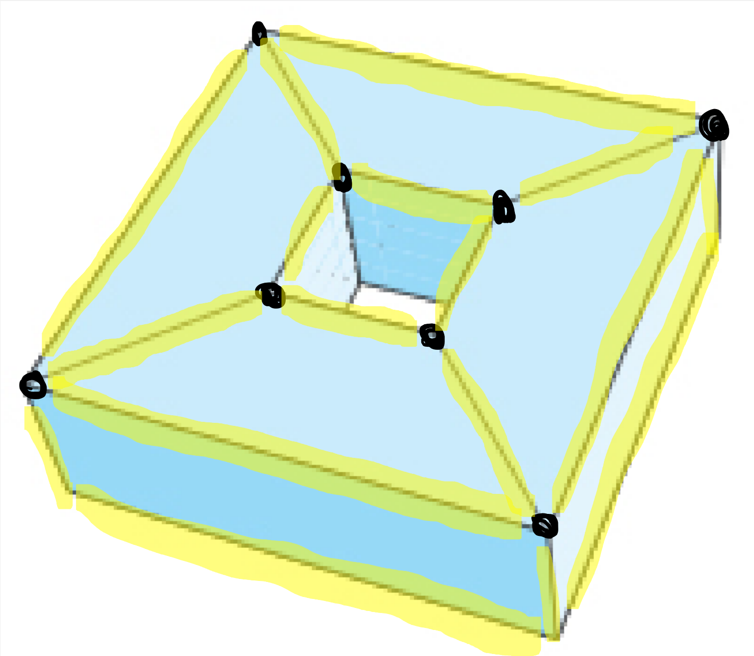


If you want to calculate V – E + F (the "Euler characteristic") of a 3D shape, then all the faces must be polygons, and they must also be **convex**: convex means that the internal angles are all less than 180 degrees.

You might need to "triangulate" the faces to make them into multiple convex parts!

# Planar graphs and Euler's formula

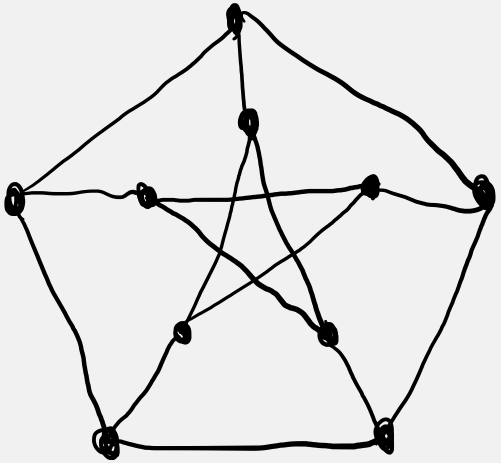
KING'S MATHS SCHOOL

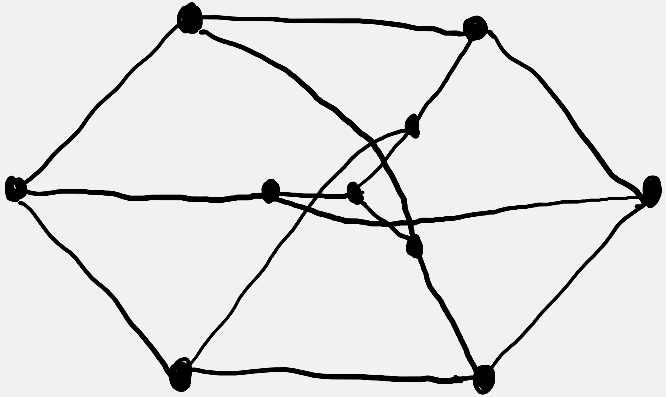


What does V – E + F = 2 give for this 3D shape?

# Other, graph-related/Euler's formula problems to look at

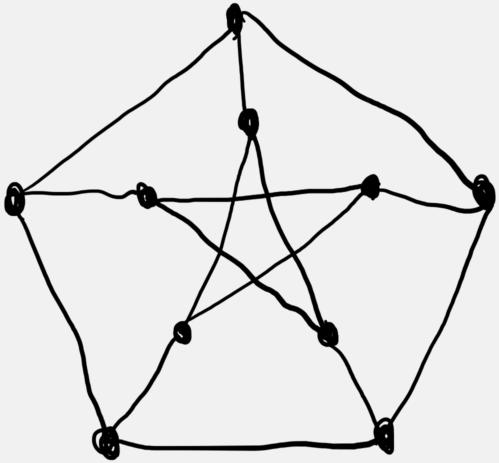
① Show that the following two graphs are isomorphic.

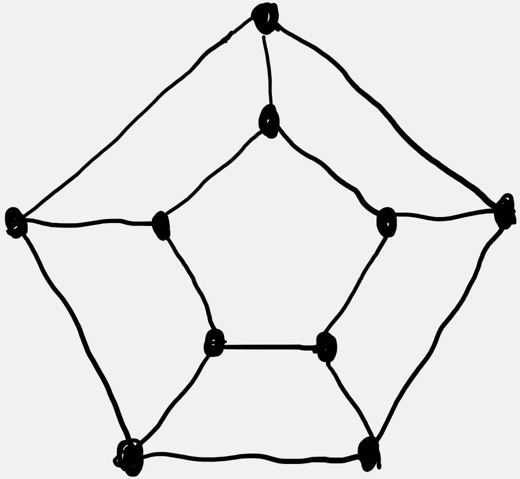




# Other, graph-related/Euler's formula problems to look at

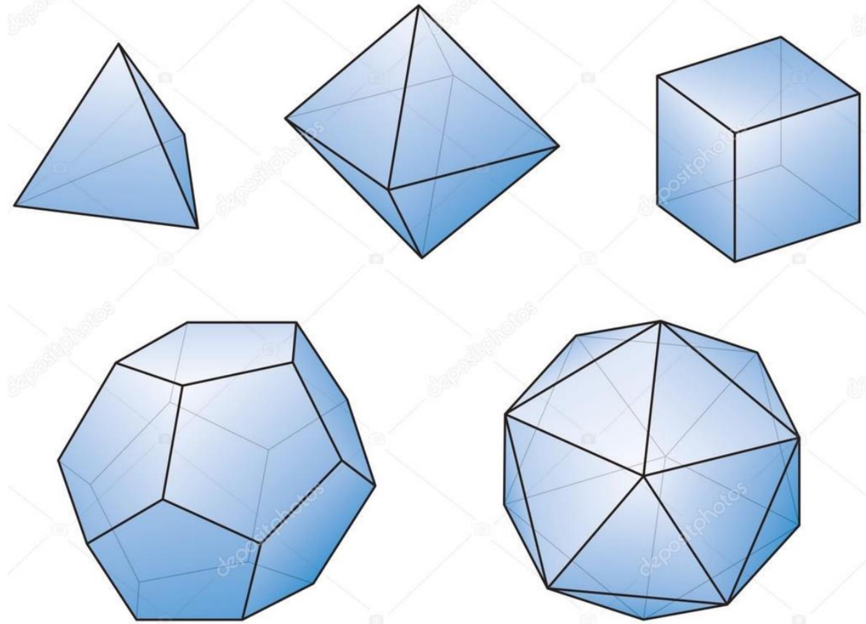
2) Show that the following two graphs are NOT isomorphic





# Other, graph-related/Euler's formula problems to look at

KING'S MATHS SCHOOL



| Name | V | E | F | n | r |
|---|---|---|---|---|---|
| Tetrahedron | 4 | 6 | 4 | | |
| Octahedron | 6 | 12 | 8 | | |
| Cube | 8 | 12 | 6 | | |
| Dodecahedron | 20 | 30 | 12 | | |
| Icosahedron | 12 | 30 | 20 | | |

**Fill in the n and r columns:**

n = number of edges on each face

r = number of edges that each vertex belongs to

# Other, graph-related/Euler's formula problems to look at

KING'S MATHS SCHOOL

Explain why $2E = nF$ and $2E = rV$.

Rearrange each of these equations to say $F = \ldots$ and $V = \ldots$, then substitute these expressions into Euler's formula to obtain $1/r + 1/n = 1/2 + 1/E$

Use this to explain why either $r < 4$ or $n < 4$ (i.e explain why we can't have $r \geq 4$ and $n \geq 4$ simultaneously). Then for each scenario find all possible solutions (remembering that n, r and E are positive integers).

This will prove that there are only five platonic solids (3D shapes where all the faces are convex regular polygons)!