General Advice.

Give yourself plenty of space:

- write one thing on each line;
- make your diagrams large enough;
- write fractions like this $\frac{3}{4}$, not like this 3 / 4 ;
- if in doubt, put in more working, not less.

Check your work by going back to the question, especially if you have used algebra. For instance, in question **17**, after deciding that there are 5 girls and 15 boys check that: the number of boys is three times the number of girls and that, after one boy is expelled and two new girls arrive, so there are 14 boys and 7 girls, that the number of boys is now twice the number of girls.

Be adventurous: if you are not sure what to do in a question, think of something you have learnt how to do in problems on this topic and try it – it might lead somewhere helpful. For instance, if there is a right angled triangle in a diagram, try using Pythagoras's Theorem, even if you can't see how it will be useful.

1 Sami buys 25 mugs for £0·84 each. He gives away 4 of them, and sells the rest at £1·40 each. What percentage profit does he make?

Solution

It's a good idea to write in words what each step of your working is calculating, and how you are going to do the calculation. put the units in at the end of each step, not in your working.

Sami pays $25 \times 0{\cdot}84 = 50 \times 0{\cdot}42 = 100 \times 0{\cdot}21 =$ £21
He sells 21 of them at £1·40 each, so gets $21 \times 1{\cdot}4 = 20 \times 1{\cdot}4 + 1 \times 1{\cdot}4 = 28 + 1{\cdot}4 =$£29·40
So he makes £8·40 profit, on his original payment of £21, so
percentage profit $= \frac{8 \cdot 40}{21} \times 100 = \frac{840}{21} = 40\%$.

2 A *Finest* cream dessert is sold in tubs of 450 ml which contain 125 ml of cream.
A *Superb* cream dessert is sold in tubs of 375 ml which contain 105 ml of cream.
Which cream dessert contains the greater proportion of cream?

Solution

Finest proportion = $\frac{125}{450} = \frac{25}{90} = \frac{5}{18}$
Superb proportion = $\frac{105}{375} = \frac{21}{75} = \frac{7}{25}$

Comparing fractions is best done by putting them over a common denominator.

Lowest common multiple of 18 and 25 is $18 \times 25 = 450$.
Proportions are: *Finest* $\frac{125}{450}$; *Superb* $\frac{7}{25} = \frac{18 \times 7}{18 \times 25} = \frac{126}{450}$
So the *Superb* dessert contains a very slightly higher proportion.

Don't forget to state your conclusion: don't leave the readers of your work to look at your calculations and decide for themselves, even if you think it's obvious.

3 Solve the equations.

a $\frac{1}{6}x - \frac{1}{4}(x-5) = 1$

b $\dfrac{x}{x-1} = \dfrac{x+1}{x-2}$

Solution

The easiest way to solve equations involving fractions is usually to multiply both sides of the equation by the simplest common multiple of the denominators.

a Lowest common multiple of 4 and 6 is 12, so multiply both sides of the equation by 12.

$$12 \times \tfrac{1}{6}x - 12 \times \tfrac{1}{4}(x-5) = 12 \times 1$$

so $2x - 3(x-5) = 12$

so $2x - 3x + 15 = 12$

so $x = 3$

b The simplest common multiple of (x – 1) and (x – 2) is (x – 1)(x – 2), so multiply both sides of the equation by (x – 1)(x – 2).

$$(x-1)(x-2)\frac{x}{(x-1)} = (x-1)(x-2)\frac{(x+1)}{(x-2)}$$

It is often helpful to put extra brackets in, as has been done here, to make sure you remember to treat the numerator or denominator of an algebraic fraction as a single factor, as you should.

so $(x-2)x = (x-1)(x+1)$

so $x^2 - 2x = x^2 - 1$

so $x = \frac{1}{2}$

In both parts of this question, it would be sensible to check your answer by putting your value of x back into the original equation.

4 The five-digit numbers 91723 and 85604 use all ten digits between them. The difference between these numbers is 91723 – 85604 = 6119.
Find two five-digit numbers which use all ten digits between them and which have the *smallest possible* difference.

Solution

To make the difference as small as possible, you want the two numbers to have first digits which differ by 1: this makes the numbers 10000 apart, but you can reduce this difference by making the number with the larger first digit have the least possible added to it from the remaining four digits, and the number with the smaller first digit have the most possible added to it from the remaining four digits.
This means that you want 0123 be the last four digits of the number with the larger first digit, and 9876 to be the last four digits of the number with the smaller first digit. this leaves 4 and 5 as the first digits, so the numbers are

50123
and 49876,
which differ by 247.

5 Find three different whole numbers A, B and C, so that

- B is the average of A and C
- A^2 is the average of B^2 and C^2.

Note that not all of the numbers can be positive.

Solution

This question is intended to be done by juggling with the numbers.
The simplest solution is probably $A = -5$, $B = 1$ and $C = 7$.

You could also tackle the question algebraically, using the equations

$$B = \frac{A+C}{2} \text{ and } A^2 = \frac{B^2+C^2}{2}.$$

The first equation gives $A = 2B - C$ which you can substitute into the second:

$$B^2 + C^2 = 2(2B-C)^2.$$

This simplifies to

$$7B^2 - 8BC + C^2 = 0$$

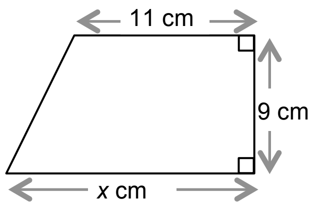
or, factorising,

$$(7B-C)(B-C) = 0$$

So $C = 7B$ (since C and B are different).

Be reassured: in the test, you would not be expected to solve this problem algebraically.

6 The area of this trapezium is 117 cm^2.What is the length x cm of its base?



Solution

You should know the formula for the area of a trapezium: (average of parallel sides) × height.

If you knew x you could work out the area of the trapezium by calculating

$$\frac{(x+11)}{2} \times 9$$

But you know this must come to 117, so you have the equation

$$\frac{(x+11)}{2} \times 9 = 117$$

so $(x+11) = \dfrac{234}{9} = 26$

so $x = 15$.

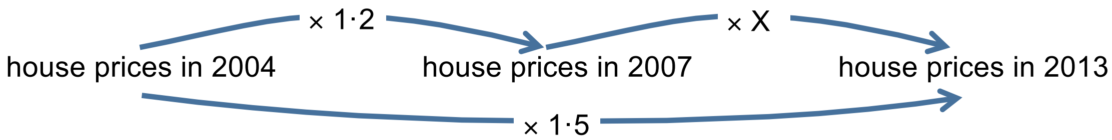
So the length of the base is 15 cm.

7 From January 1st 2004 until January 1st 2013, average house prices in London rose by 50%. From January 1st 2004 until January 1st 2007, average house prices in London rose by 20%. By what percentage did average house prices in London rise from January 1st 2007 until January 1st 2013?

Solution

Questions involving several percentage changes are usually best done by using a scale factor.

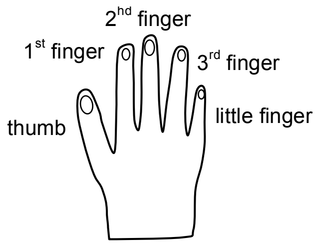
A rise of 50% is calculated by multiplying the original amount by 1·5; a rise of 20% is calculated by multiplying the original amount by 1·2.



Let the scale factor for the rise in prices from 2007 to 2013 be *X*.
The diagram shows why multiplying by 1·2, and then by *X*, is the same as multiplying by 1·5,
so $1{\cdot}2 \times X = 1{\cdot}5$

so $X = \dfrac{1{\cdot}5}{1{\cdot}2} = 1{\cdot}25$, which means that house prices rose by 25% from 2007 to 2013.

8 A boy counts on his fingers, backward and forwards across his right - hand as follows: thumb, 1st finger, 2nd finger, 3rd finger, little finger, 3rd finger, 2nd finger, 1st finger, thumb, 1st finger, … and so on.



If he starts counting at one, on his thumb, which finger will he be on when he reaches two thousand and thirteen?
Explain clearly how you decided.

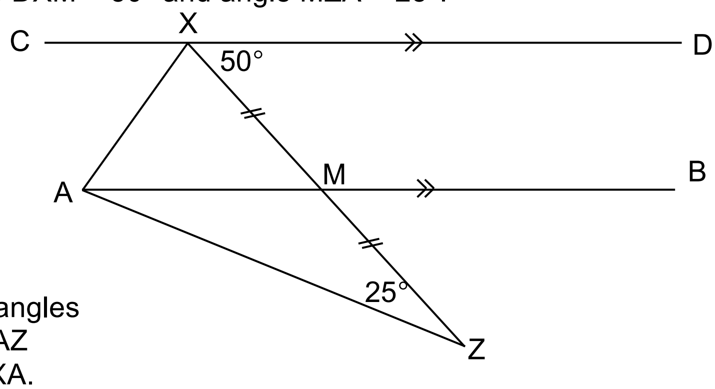
Solution

"Explain clearly", in this question, means you cannot just show a calculation, but you need to say in words why that calculation gives the answer you want.

The boy counts his thumb on 1, 9, 17, and so on: that is on numbers that are one more than a multiple of eight.
The last multiple of eight before 2013 is 2008, so the boy will count on his thumb on 2009.
Therefore he will count on his
  1st finger on 2010
  2nd finger on 2011
  3rd finger on 2012
and little finger on 2013.

An answer without a suitable explanation might look like this.
  2013 = 8 × 251 + 5 so little finger.

9 In the diagram line CD is parallel to line AB, and M is the midpoint of line XZ.
Angle DXM = 50° and angle MZA = 25°.



Find angles
i MAZ
ii CXA.
showing and explaining each step in your working.

Solution

In geometry problems, when you are asked to explain your working, write each statement on a new line, with a reason why the statement is true (usually in brackets afterwards).

Angle BMZ = 50° (corresponding with angle DXM on parallel lines CD and AB).
So angle AMZ = 180 – 50 = 130° (angles on a straight line add to 180°).
So angle MAZ = 180 – 130 – 25 = 25° (angles in a triangle add to 180°).
This means that triangle ZMA is isosceles (base angles MAZ and MZA are equal).
So AM = MZ (sides opposite base angles of an isosceles triangle).
So AM = XM (it is given that M is the midpoint of XZ, so MZ = XM).
So triangle XMA is isosceles (two equal sides).
So angle MAX = angle MXA (base angles of isosceles triangle MAX).
But angle XMA = 50° (alternate with angle DXM on parallel lines CD and AB).
So angle MAX = $\frac{1}{2}(180-50) = 65°$ (angles in a triangle add to 180°).

10 You are given that *N* is a whole number which is a multiple of 6 and a multiple of 10.
For each of the following statements say whether you can be sure it is true, it might or might not be true or it definitely isn't true. Explain your answer in each case.
a *N* is a multiple of 60.
b *N* has a factor of 15.
c *N* is a factor of 100.
d *N* is a multiple of 7.

Solution

*Notice in the solution that follows how you need a **justification** for saying that the statement is always true or always false: just giving **examples** to show that it can be true or false is not enough, because that only shows that the statement is sometimes true or false. However, two examples **are** enough to show that a statement might or might not be true.*

If a number is a multiple of 6, it has factors of 2 and 3; if it is a multiple of 10, it has factors of 2 and 5: so the number must have factors of 2, 3 and 5, so it must be a multiple of 30.

This means that **b** must be true, and **c** must be false.

a might or might not be true: 90 is a multiple of 6 and 10, but not a multiple of 60, but 120 is a multiple of 6 and 10, and a multiple of 60.

d might or might not be true: 30 is a multiple of 6 and 10, but not a multiple of 7, but 210 is a multiple of 6 and 10, and a multiple of 7.

11 In a school, there are 47 pupils in year ten.
Nine girls are in the netball squad and twelve girls are in the hockey squad. Three girls are in both squads and seven girls are in neither squad.
Thirteen boys are in the football squad and eight boys are in the basketball squad. Five boys are in neither squad. How many boys in year ten are in the football squad and the basketball squad?

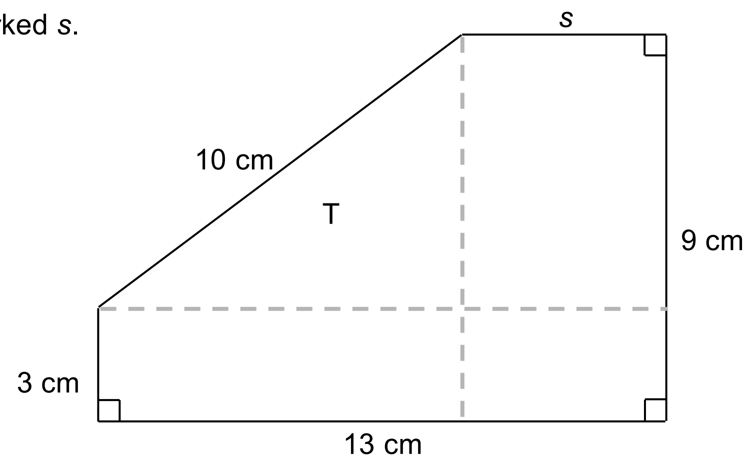
Solution

Here, you need to realise that the total number of girls in the netball and hockey squads is not 9 + 12, because 3 of the girls have been counted twice in that addition. So the total number of girls in the netball and hockey squads is 9 + 12 – 3 = 18. Since there are 7 girls in neither squad, there are a total of 25 girls in year 10.

Therefore there are a total of 22 boys in year 10. Because 5 of these are in neither squad, 17 boys are in the football and basketball squads. But 13 + 8 = 21, so 21 – 17 = 4 boys must have been counted twice in that addition: there are 4 boys in both squads.

12 The diagram shows an irregular pentagon. The lengths of four of the sides are shown in the diagram. Three of the angles in the pentagon are right angles, as shown.

Find the length of the side marked *s*.



Solution

The dashed lines have been added to the diagram. Right angled triangle T has been marked.

You can see that the height of the right angled triangle is 9 – 3 = 6 cm,
but its hypotenuse is 10 cm, so by Pythagoras's Theorem its base is

$$\sqrt{10^2 - 6^2} = \sqrt{64} = 8 \text{ cm.}$$

This means that s = 13 – 8 = 5 cm.

13 A man is going to ride his horse from his house to a town 39 miles away. After he leaves his house, he rides at 24 miles per hour for the first forty-five minutes, then speeds up when he reaches a good track and rides at 36 miles per hour for the next 15 miles, until he reaches a river. He then lets the horse drink for ten minutes. After this, he continues to town at a gentle pace through the woods. Altogether, he finds he has taken two hours to travel the 39 miles from his house to the town. What is his average speed when he is riding at a gentle pace through the woods?

Solution

*As in question **1**, it's a good idea to write down what each step of your working is calculating, because this is a complicated question, and it's easy to lose track of what you are doing. In this case, making a table of what you know, and updating it as you do each calculation, is sensible.*

This table shows the initial information you are given: notice how all the figures have been converted to compatible units: if the distance is in miles and the speed in miles per hour, you want the times in hours, not minutes.

	first section	second section	rest	third section	whole journey
distance (miles)		15	0		39
speed (miles per hour)	24	36	0		
time (hours)	$\frac{3}{4}$		$\frac{1}{6}$		2

The distance for the first section is speed × time = $24 \times \frac{3}{4} = 18$ miles,

and the time for the second section is $\frac{\text{distance}}{\text{speed}} = \frac{15}{36} = \frac{5}{12}$ hour (which is 25 minutes).

Now the table looks like this

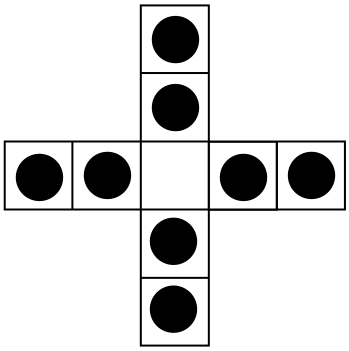
	first section	second section	rest	third section	whole journey
distance (miles)	18	15	0		39
speed (miles per hour)	24	36	0		
time (hours)	$\frac{3}{4}$	$\frac{5}{12}$	$\frac{1}{6}$		2

So we can calculate, for the third section, distance = 39 – 18 –15 = 6 miles,
and time = 120 – 45 – 25 – 10 = 40 minutes = $\frac{2}{3}$ hour.

The speed in the third section is therefore

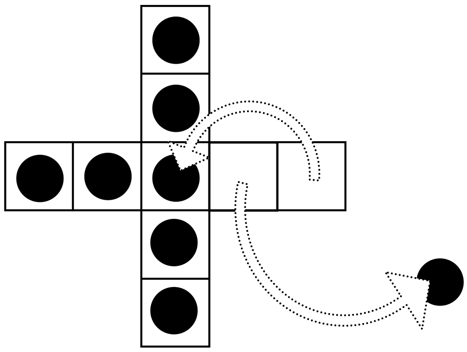
$$\frac{\text{distance}}{\text{time}} = \frac{6}{\frac{2}{3}} = 6 \times \frac{3}{2} = 9 \text{ miles per hour.}$$

14 The diagram shows nine squares. There are counters on eight of the squares; the ninth square is empty.



A move on this diagram consists of jumping one counter over another to an empty square. The jumping counter moves in a straight line (up, down, left or right) and lands exactly two squares away from its original position. The counter that is jumped over is removed.

For instance, after the first move the position could be



a Show that all four possible first moves lead to a rotation or reflection of the same position.

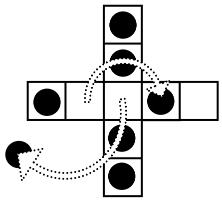
The game ends when it is impossible to make a move.

b Show that, however you continue this game, there are at least two counters left when the game ends.

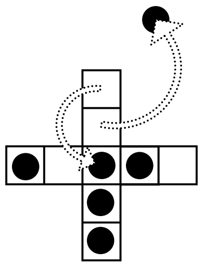
Solution

a The only four possible first moves are to jump one of the counters at the end of an arm to the centre: that arm is then empty, and all other squares have counters on, which is a rotation of the second diagram shown above.

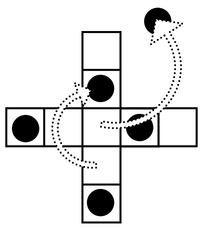
b You only need to consider one of the possible positions after the first move, because the others are just rotations of it.
The only possible second move is like this:

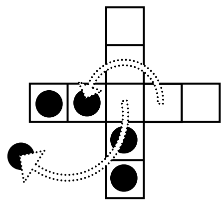


After the third move, you must be in the following position, or a rotation or reflection of this position.

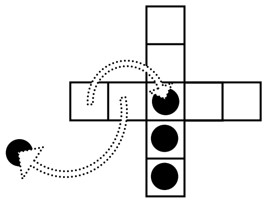


Now you have a choice: after the fourth move you could be in either of these positions, or rotations or reflections of them.

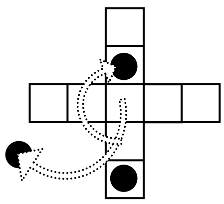




In the first case, no further moves can be made, but in the second case, after a fifth move, you would be in the following position, or a rotation or reflection of it.



Finally, after a sixth move, you could only be in the following position, or a rotation or reflection of it.



The only two possible outcomes leave two, or four, counters on the board, so always at least two.

When you are trying to show what must happen as you follow a set of rules, it is important to follow up all the possibilities at each stage: don't just check what you think is the best thing to do.

15 To **double** a number means to multiply it by two.
In this question, to **twiddle** a number will mean to add four to it, and to **flip** a number will mean to subtract it from 8 (so **flip**ping 3 gives 5, **flip**ping –1 gives 9 and so on).

a If you **twiddle** a number and then **twiddle** the answer, the overall effect of the two operations is to add eight to the number. What would be the overall effect of the two operations if you:

i **flip** a number and then **flip** the answer?

ii **twiddle** a number and then **flip** the answer?

iii **flip** a number and then **twiddle** the answer?

b Show that if you **twiddle** a number, **flip** the answer and then **twiddle** the answer to that, this has the same overall effect as just **flip**ping the number once.

c Find a sequence of three operations, each a **twiddle** or a **flip**, that you can carry out on any number and which has the overall effect of just changing the sign of the number.

d **i** Find a sequence of six operations, each a **twiddle** or a **flip**, that has the overall effect of leaving every number you apply it to unchanged.

ii Is it possible to find an odd number of operations, each a **twiddle** or a **flip**, that has the overall effect of leaving every number you apply it to unchanged? Justify your answer.

Solution

a **i** The number stays the same.

ii The overall effect is the same as subtracting the number from 4.

iii The overall effect is the same as subtracting the number from 12.

b If the starting number is x, twiddling it gives $x + 4$, flipping this answer gives

$$8 - (x + 4) = 4 - x$$

and twiddling this answer gives

$$(4 - x) + 4 = 8 - x,$$

which is the same outcome as just flipping x once.

c twiddle, followed by twiddle, followed by flip has the outcome

$$8 - ((x + 4) + 4) = - x$$

when applied to x.

d **i** twiddle, twiddle, flip, twiddle, twiddle, flip starting with x has the outcome

$$-(-x) = x$$

when applied to x.

ii (A) Every combination of flip and twiddle has one of the following outcomes when applied to x:

$$x + N,\ N - x$$

where N is an integer (which is always a multiple of 4).

(B) The first form arises when there are an even number of flips, and the second when there are an odd number of flips.

(C) Therefore, to get an outcome of x you need an even number of flips.

(D) But then, to get an odd total number of operations, there must be an odd number of twiddles.

(E) But each twiddle either adds or subtracts 4 from N, so an odd number of twiddles cannot take you from N being zero back to N being zero.

16 Five bananas and two kiwi fruit cost £2·30, and four bananas and three kiwi fruit cost £2·47. How much would it cost to buy two bananas and one kiwi fruit?

Solution

In algebra problems, it is a good idea to state clearly at the start of the question what x *stands for.*

Let x be the cost in pence of a banana, and y the cost of a kiwi fruit in pence.
Then, working in pence throughout,

$$5x + 2y = 230$$
$$4x + 3y = 247$$

Multiplying the first equation by 3, and the second by 2, gives

$$15x + 6y = 690$$
$$8x + 6y = 494$$

and subtracting the equations then gives

$$7x = 196$$

so $x = 28$ and

$$5 \times 28 + 2y = 230$$

so $y = 45$.
A banana costs 28p and a kiwi fruit 45p.

17 At the start of the year, there were three times as many boys as girls in a class. At Christmas, two new girls joined the class and one of the boys was expelled. Then there were only twice as many boys as girls in the class. How many boys and girls were there in the class at the start of the year?

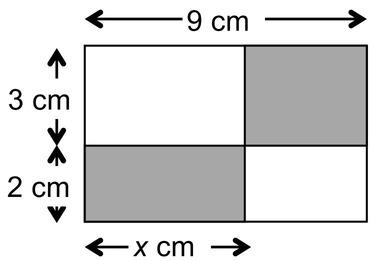
Solution

Let x be the number of girls in the class at the start of the year.
Then there are $3x$ boys in the class at the start of the year.
After Christmas, there were $x + 2$ girls and $3x – 1$ boys in the class.
So

$$3x - 1 = 2(x + 2)$$

so $x = 5$.
There were 5 girls and 15 boys in the class initially.

18 The two shaded rectangles have equal area. What is the total shaded area?



Solution

Let x cm be the width of the lower shaded rectangle, as shown on the diagram.
Then the width of the upper shaded rectangle is $(9 – x)$ cm
So, because the areas are equal,

$$2x = 3(9 - x).$$

so $x = \frac{27}{5}$ (= 5·4 , if you like).

So the shaded area = $2 \times \left(2 \times \frac{27}{5}\right) = \frac{108}{5}$ cm^2

19 PROBLEM Find x, y and z if

$$x + y = 5$$
$$z + y = 11$$
$$x + z = 13.$$

METHOD The first equation can be rearranged to

$$x = 5 - y.$$

and the second equation can be rearranged to

$$z = 11 - y.$$

These two rearrangements imply that

$$z + x = (5 - y) + (11 - y) = 16 - 2y$$

but the third equation says that

$$z + x = 13,$$

so

$$16 - 2y = 13.$$

This is a straightforward equation which can be solved to give $y = 1\frac{1}{2}$.

The first step of this METHOD can now be used to conclude that

$$x = 5 - y = 5 - 1\tfrac{1}{2} = 3\tfrac{1}{2}$$

and

$$z = 11 - y = 11 - 1\tfrac{1}{2} = 9\tfrac{1}{2}.$$

a Use the METHOD above to solve the PROBLEM

Find x, y and z if

$$x + y = 12$$
$$z + y = 17$$
$$x + z = 21$$

b Adapt the METHOD to solve the PROBLEM

Find x, y and z if

$$x + 3y = 14$$
$$z + 5y = 21$$
$$2x - z = 5$$

c Try to adapt the METHOD to solve the PROBLEMs below. What goes wrong?

i Find x, y and z if

$$x - y = 15$$
$$z - y = 6$$
$$x - z = 11$$

ii Find x, y and z if

$$x - y = 15$$
$$z - y = 6$$
$$x - z = 9$$

d One of the PROBLEMs in **c** has many solutions, even though the METHOD does not work; but the other PROBLEM has no solutions.
Identify which problem is which, and explain what the difference is between the two PROBLEMs which means that one has solutions and the other does not.

Solution

a $x = 12 - y$
$z = 17 - y$
so $(12 - y) + (17 - y) = 21$
so $y = 4$
and $x = 8$ and $z = 13$.

b $x = 14 - 3y$
$z = 21 - 5y$
so $2(14 - 3y) - (21 - 5y) = 5$
so $28 - 6y - 21 + 5y = 5$
so $y = 2$
and $x = 8$ and $z = 11$.

c **i** $x = 15 + y$
$z = 6 + y$
so $(15 + y) - (6 + y) = 11$
so $15 + y - 6 - y = 11$
so $9 = 11$
which it isn't.

ii $x = 15 + y$
$z = 6 + y$
so $(15 + y) - (6 + y) = 9$
so $15 + y - 6 - y = 9$
so $9 = 9$
which it is, but that doesn't tell you what y is.

d The first problem has no solutions, but the second has many solutions (for instance, $x = 9$, $y = -7$, $z = -1$ or $x = 18$, $y = 3$, $z = 9$).

In the first set of equations,
$x - y = 15$
$z - y = 6$
$x - z = 11$
adding the second and third equation gives $x - y = 17$, but the first equation says that $x - y = 15$: it is impossible for $x - y$ to have two different values, so there is no solution.

In the second set of equations,
$x - y = 15$
$z - y = 6$
$x - z = 9$
adding the second and third equation gives $x - y = 15$, and the first equation says that $x - y = 15$, so there is no problem. In fact, any choice of y value will give a solution of the equations, provide $x = y + 15$ and $z = y + 9$.